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[illegible]

By [illegible]

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SELECT ENGLISH POETRY.

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THE

SCIENCE OF ARITHMETIC:

A SYSTEMATIC COURSE OF

NUMERICAL REASONING & COMPUTATION,

WITH VERY NUMEROUS EXERCISES.

BY

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LONDON:

SIMPKIN, MARSHALL, & CO., STATIONERS' HALL COURT;

HAMILTON, ADAMS, & CO., PATERNOSTER ROW.

EDINBURGH: OLIVER AND BOYD; W. P. KENNEDY.

1855.

LONDON:
J. & W. RIDER, PRINTERS, BARTHOLOMEW CLOSE.

PREFACE.

This book differs from others bearing a similar title in several important particulars.

I. The investigation of the principle on which a rule of Arithmetic depends always precedes the statement of the rule itself.

II. Every process employed in the solution of questions is referred to some general law or truth in the theory of numbers. Thus the operation of "carrying" is explained in § 32, which illustrates the axiom, that—We do not alter the value of any quantity, if what we take from one part we add to another.

III. Such general truths are all distinctly stated and printed in italics. If self-evident they are illustrated by simple numerical examples; if otherwise, short demonstrations are added; and in every case the truth itself is enunciated in a concise symbolical form. So, the method of equal additions, which is employed in the ordinary mode of working Subtraction, is explained numerically and otherwise,* and shown to be dependent on the axiom, that—We do not alter the difference between two unequal quantities if we add equal sums to both. And this truth is again expressed symbolically in the following way:—

General Formula. If $a - b = x$; then
$$(a + c) - (b + c) = x.$$

* §§ 44 and 45.

IV. The theory of decimals, and rules for the solution of money questions by the decimal method, are placed earlier in the course than usual, and are thus made available throughout the rules of Proportion and Interest. The proposed change in our coinage is fully described,* and special attention has been bestowed upon this part of the work,† under the conviction that an accurate and ample knowledge of the decimal system is more than ever important at this moment, whether the decimalization of our money, weights, and measures takes effect immediately or not.

V. The logical relations of the several parts of Arithmetic are clearly marked by their arrangement. For example, Reduction is not treated as a separate rule, but so much of it as belongs to Multiplication falls under that head, while the rest takes its proper place as one of the practical applications of Division. Interest and Discount, and the kindred rules, are grouped together as illustrations of the doctrine of Proportion; and Practice is treated as a branch of Fractional Arithmetic.

VI. The tables of Foreign Currency and of English Weights and Measures, are accompanied by an explanation of the origin of the several standards in common use, and of the causes which have led to their diversities and irregularities.

Many important advantages would accrue to beginners as well as to advanced students, if Arithmetic were regarded more as a branch of mathematical science, and less as a mere system of practical rules. The art of computation is undoubtedly of much value in the business of

* §§ 296—303. † Pp. 140—179.

life; but the habit of investigating the principles on which this art is based is not of inferior importance. The first gives to the student a mastery of figures which will be serviceable in commercial and scientific pursuits; the second tends to concentrate his attention; to induce habits of patient abstraction and accurate thought; to familiarize him with the laws of reasoning, and to compel him to examine well the grounds of every inference he draws. Such habits as these will be invaluable in *every* pursuit and duty of life, for they will help to make him a sounder and more modest reasoner, and therefore a wiser man.

The value of the exact sciences as instruments of mental discipline has long been recognized. To omit them from any scheme of instruction, however humble, is to allow an important class of the mental faculties to remain untrained. In the limited curriculum of our common day schools, Arithmetic holds a place analogous to the Mathematics of a University course. It is the only one of the pure sciences usually admitted into such a school, and the only instrument there available for severe and systematic logical training. To degrade Arithmetic into a mere routine of mechanical devices for working sums, is, even in a school for young children, to commit as grave and mischievous a mistake as if our University professors were to permit the rules of mensuration to supersede the study of Euclid, or to displace the rigid analysis of the calculus and the higher trigonometry, in order to make room for land surveying, the rules of navigation, or the construction of tide tables.

It is only when looked at in this higher aspect that Arithmetic can become an efficient instrument for disciplining the judgment and improving the mental powers; indeed, it has no right to be called a science at all,

so long as it is limited to ciphering on a slate, and does not include a systematic acquaintance with *principles* as well as rules. To promote such a knowledge of principles, something more is necessary than a theoretical treatise on the one hand, or a book of rules, with explanations appended, on the other. Throughout this work, therefore, the principle is in every case first explained and illustrated, and then the rule is shown to follow from it naturally and necessarily. For example, the rules for the extraction of the square and cube roots are made to depend entirely upon the theory of Involution. When a pupil has become familiar with the law for the formation of the second or third power,* he will scarcely need to be told the rule for the extraction of either of those roots; while on the other hand, without an acquaintance with that law the rule must ever appear to him to be arbitrary and unmeaning.

Besides the explanations necessary to aid the comprehension of the ordinary rules, the plan of this work comprises special exercises on the properties of numbers, demonstrations of the abstract propositions of Arithmetic in a form adapted for repetition by the student,† and questions at the end of each chapter, intended especially to test the pupil's knowledge of the theory of numbers. It will also be observed, that one example at least is appended to each rule, in which every part of the process is analyzed and the separate value of every line of figures clearly shown. These features of the work will, it is hoped, be of use as suggesting some new and not unprofitable methods of exercising the minds of pupils in connexion with this study. For example, a sum when finished

* As enunciated in p. 238, *et seq.* † See Division of Fractions, p. 120, *et passim*

may occasionally be taken as the subject of examination and analysis, in the same manner as a sentence in grammar is used for parsing and construing. The exact meaning and value of every figure should be investigated. Every process should be justified by reference to some axiom or proposition, in the same way as a grammarian would refer every detail in the construction of a sentence to some definite rule; and if the student be sufficiently advanced, it is desirable, by way of giving neatness and finish to his knowledge of a truth, that he should be able to recognize and interpret it when he meets it in the symbolical form.

Although the main object of the authors has been to furnish a systematic and coherent exposition of the theory of Arithmetic, yet the practical uses of the Science have been scrupulously kept in view. A comparison of this work with those which are specially valued on account of the abundance and character of the examples, will show that it comprehends everything that is usually regarded as practical or valuable in Arithmetic, in addition to those reasonings and formulæ by which the theory of numbers has been elucidated. The examples have been chosen with especial reference to the pursuits of a commercial people; the subjects of statistics, insurance, stocks, and interest have received particular attention, and an unusual number of business questions occur both throughout the work and in the Appendix.

Some explanation may seem necessary of the word "axiom," as used at the head of some general propositions and not of others. Without attempting to determine how far even the simplest truths of science are axiomatic and independent of experience, it will suffice to say that the word has been generally applied to such numerical theo-

rems as seemed self-evident from the terms in which they were enunciated, and were not to be deduced as inferences from any previous statement in the work.

Many of the questions are selected from the Cambridge and London examination papers, and from those proposed to schoolmasters who have been candidates for certificates of merit. The student who masters the reasonings and becomes familiar with the rules of this book, will, as far as Arithmetic is concerned, be competent to pass with credit the ordinary examination for the degree of B.A. at either of the Universities.

It only remains to caution students against attempting to make a very hasty progress through this work. The sort of proficiency in Arithmetic which is obtained by evading its difficulties and hurrying on to the advanced rules, is very worthless. It is fatal to studious habits and is not even available for practical purposes. A few principles thoroughly sifted and understood, will be found to form a better substratum for future mathematical or commercial attainments than all the rules of a book, if studied apart from those principles. To those departments of this subject which especially require thought and examination, Lord Bacon's aphorism applies with remarkable force:—"Non inutiles scientiæ existimandæ sunt, etiam quarum in se nullus est usus; si ingenia acuant et ordinent."

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APPENDIX.

The Sections which are distinguished by a double asterisk (**), occupy their proper positions in the book, as far as their logical relation to the rest determines it, they are nevertheless not absolutely necessary for the comprehension of the subsequent rules, and may be omitted by a student who is reading the work for the first time.

THE SCIENCE OF ARITHMETIC.

1. Arithmetic is the science of number.*

The first and simplest notion we must obtain in the science of Arithmetic is that of UNITY, or oneness.

2. Any object, or any magnitude considered singly, and without reference to its parts, is called a Unit.

Thus when we perceive and speak of one man, a horse, or a stone, we have before us the idea of unity, and are said to be thinking of a unit.

3. Any collection of units of the same kind is called a Number.

When we speak of a crowd of persons, a heap of stones, or ten books, or five hours, we have before us the notion of number. These expressions, however, would not be understood if the notion of *one* person, *one* stone, *one* book, or *one* hour was not in the mind first. Unity, therefore, is the one magnitude with which all magnitudes of the same kind are compared.

4. If we look at a row of trees or a group of children, we cannot help considering one tree and one child as the unit or standard of reference, when we attempt to number them; but in expressing the length of a line, or the size of a field, or the duration of time, we are left to choose any unit we please. Thus the same length of

* It forms a branch of the Mathematics, which also include Algebra, Geometry, Trigonometry, and several other sciences. The first of these investigates the same subject as Arithmetic, but in a wider sense. The remaining two investigate the properties of space. All the branches of Mathematics relate to magnitude and the modes of measuring it.

line may either be expressed as two yards or six feet: the number used being dependent on the sort of length which has been selected as the unit. In any magnitude of this kind, which is capable of *continuous* increase, such as time, space, length, or weight, the choice of the unit is arbitrary. But when we use number to describe collections of separate objects of the same kind, one of those objects is always taken as the unit.

The least instructed person who looks at the heavens and sees six bright stars, and also into a garden and sees six roses, cannot help understanding that although there is no resemblance between a star and a rose, yet there is some resemblance between the two notions of the *six* stars and the *six* roses. These two compound ideas have something in common, and that something is the idea of number, which is expressed in our language by the word six.* It is not so expressed by all nations, but, however language may differ, the thought is universally the same.

5. The science of Arithmetic is intended to give clearness and accuracy to our notions about number. We may think about number without the aid of Arithmetic, but always in a vague and indistinct way, as when we see two heaps or collections of objects, and observe that the one contains *many* and the other *few*, or that the one is a *larger* heap than the other; but when we wish to ascertain *how* many or *how* few either contains, or *by how much* the size of the one exceeds that of the other, we are compelled to employ the symbols and the processes of ARITHMETIC.

* When we think of number in connexion with objects, or when, in using a word which represents number, we also use the word which describes the kind of magnitude to which it refers, the number is said to be *concrete*. But when, as in the case just mentioned, we *withdraw* or abstract the thought of *six* from that of the objects to which it refers, and think of six, eight, or ten units, without reference to the kind of unit to which they may be attached, the number so considered is an *abstract* number.

In like manner, the notion of a thing associated with a quality which belongs to it is a compound or *concrete* notion, the notion of the quality alone is an *abstract* notion. From the two very dissimilar objects, a blue sky and a blue piece of paper, the mind can withdraw the thought of the quality alone, and think of *blueness*. This is an abstraction, a quality considered by itself. So if after seeing several bright things, we consider *brightness* by itself, or after hearing several pleasant sounds, we consider the separate quality of *pleasantness*, we have exercised the faculty of abstraction, and the words we thus use are called *abstract* nouns.

LANGUAGE OF ARITHMETIC.

SECTION I.

6. The language employed in Arithmetic may be considered in two ways :—I. As consisting of a set of words or sounds intended to express number; and II. As composed of written characters or signs.

Some rude nations have the former without the latter; they have a language about number which can be addressed to the ear, but have no figures or symbols which can be presented to the eye.

7. Numeration is the art of numbering, or of expressing number *in words*.

Observation.—The various collections of units which we meet with and wish to express are so numerous, that if we had a name for every such collection, Arithmetic would require more words than all the sciences put together. There is in fact no limit to our power of making new numbers, for however great may be the collection of units which one person may speak of, another person may speak of a greater, and may conceive an infinite number of additions to be made to it. All nations, therefore, have a few separate words or names for some particular collections of units, and express all other numbers by putting together these words in different ways.

8. We have in our language fifteen distinct words only, viz.: *one*, which means the unit; *two*, *three*, *four*, *five*, *six*, *seven*, *eight*, *nine*, *ten*, *eleven*, *twelve*, *hundred*, which means ten taken ten times; *thousand*, which means one hundred taken ten times; *million*, which means one thousand taken one thousand times.

All other numbers or collections of number are expressed by combining some of these words. The words billion, trillion, &c., though found in books, are not generally used.

All systems of numeration are founded on the truth of the following axiom.

9. Axiom I.—*All the parts of a number put together make up the whole in whatever order they may be taken.*

Demonstrative Example.—The first word in our language which illustrates this principle is *thirteen;* taken to pieces it means *three* and *ten*. We might have called it ten-three, or nine-four, or seven-

six, for in each of these cases we should have spoken of two parts which together make up the whole; or six-five-two, or three-four-six, or by any other name which mentioned all its parts. But it is more convenient to consider all numbers divided in the same way.

10. The number ten has been chosen as the basis of all our calculations, and we consider all numbers as made up of tens or some collection of tens. Our system is therefore called DECIMAL.*

It is probable that this practice, which prevails in nearly all parts of the world, arose from the use of the ten fingers in simple calculations. In counting on the fingers (as children often do now) we can go on easily until we come to ten, but are obliged to use some other contrivance for all higher numbers. If we had had twelve fingers instead of ten we should probably have acquired the habit of counting by twelves, and considering all large numbers as composed of so many twelves.

It would have been just as easy, had we been accustomed to it, to consider numbers as made up of fours, or nines, or in any other way than by tens: habit alone makes our present method seem simple. If, for example, we hear of a number consisting of nine, seven, and eight, we have not at first a clear notion of what it means, but we add them together and call them twenty-four, and then we understand how many are spoken of. Yet it is not now before the mind as a whole, but in three parts (two tens and four), and it was in three parts before (nine, seven, and eight). But of all the possible methods of considering the number, that which breaks it up into tens is the easiest only because we are accustomed to it.

Thus fourteen, fifteeen, and nineteen, mean four and ten, five and ten, and nine and ten; *twenty* means two tens; *thirty*, three tens; and *ninety*, *nine* tens; seventy-nine means seven tens and nine units; two hundred and thirty-six means two of the collections called hundreds (*i.e.*, two tens of tens), three tens, and six units.

EXERCISE I.

Write out in other words what is meant by the following expressions:—

* From *decem*, ten, Latin.

Six-teen; twenty-seven; eighty; forty-five; three hundred and sixty; five thousand four hundred and seventy-four; five hundred and ninety-five; fifty-eight; twelve; seventy-nine; one million six hundred thousand; twelve hundred and ninety; three thousand and forty-five; eight hundred and sixty; three millions; eight thousand four hundred and sixteen; seventy thousand and fourteen.

11. Notation is the art of expressing numbers in *written* characters.

The written language, like that which we speak, uses a few symbols to express small numbers, and collects some of these together in a certain order to express larger ones. We have not so many separate figures or characters as we have separate words. The only signs in use are the following: 1 means one; 2, two; 3, three; 4, four; 5, five; 6, six; 7, seven; 8, eight; 9, nine.* When any one of these figures is found alone, and in this form, it only stands for one of the first nine collections of units, and is said to indicate *unity of the first degree.* All numbers above nine are, however, to be represented by the same figures; so sometimes we want to make 1 mean one ten; 2 to mean two tens; 3, three tens; and so on. Whenever the nine digits are used in this sense they are said to indicate *unity of the second degree.* At other times we require to make use of the figures 1, 2, 3, &c., to mean 1 hundred, 2 hundred, &c.; and in this case they refer to *unity of the third degree.* When they are employed to signify thousands, they are *units of the fourth degree.* We therefore want some contrivance to show when the figures are used in one sense and when in the other.

* It will be observed that our written language is not quite so copious as the spoken, and does not exactly correspond to it. Thus the word *eleven* is a simple name which to the ear suggests no thought of combination, but the figures 11 represent composition, and inform us that the number consists of two parts, viz., one collection of units and one. In French, the difference between the written and the spoken language is much wider. The number *ninety-two* is written in that language as in English (92), and represents 9 collections of ten each and 2 units. But it is called by a word (quatre-viugt douze) which signifies 4 twenties and twelve. Many other of the words used to express number in the French language are founded in like manner on a division of the whole into twenties, while the written characters treat all collections of units as consisting of tens.

12. Suppose we used the simple digit when we wished it to represent unity of the first degree, and placed a mark over it thus ($'$) when we wished it to stand for tens, then the expression $17'$ or $7'1$ would mean 1 and 7 tens, or 7 tens and 1. Either form would serve the purpose. So $6'5$ would mean 6 tens and 5, and would represent the same number as $56'$. In like manner we might represent unity of the third degree, or so many *tens* of *tens*, by a digit with 2 strokes above it; thus $5''$ would mean 5 tens of tens, or 5 hundreds; thousands, or tens of hundreds, would be expressed by digits with 3 strokes above them ($6'''$), and so on in like manner for all higher numbers. If such an arrangement existed the expression $4'''6''82'$ would mean 4 units of the fourth degree (4 thousands), 6 units of the third degree (6 hundreds), 8 units of the first degree (8), and 2 of the second (2 tens). As the value of each figure would then be known by the number of marks above it, the digits might be put down in any order without altering their meaning. Thus $2'4'''86''$, or $84'''2'6''$, would represent the same collections of numbers only in a different form.

13. Another method of representing larger numbers by the help of the nine simple figures might be to use the letters A, B, C, &c., to show the different meanings of the digits. Thus 7A might mean 7 tens, 7B seven hundreds, 7C seven thousands, and so on. On this plan the expression 8B, 6D, 5, 4A, would mean 8 units of the third degree (8 hundreds), 6 of the fifth (6 tens of thousands), 5 of the first (5), and 4 of the second (4 tens). The number which we call 5 thousand 4 hundred and 9 tens and 4, would be written 5C, 4B, 9A, 4, or 4B, 4, 9A, 5C. For as each figure's value would be shown by the sign which followed it, they might be placed in any order without altering the meaning of the whole expression.

Such expressions as 2^3, 4^2, 7^5, could easily have been employed to signify 2 units of the third degree, 4 units of the second degree, and 7 units of the fifth degree. We use these forms now with another meaning (see Multiplication), but they would have served equally well for this purpose.

Many other plans might be devised for representing any numbers whatever on the decimal system, by the help of nine digits or significant figures; but the most convenient is that which is in common use.

14. The meaning of each separate figure used in our Arithmetic is shown by its POSITION only.

For example, that which in our spoken language is called ten, is written as the figure 1, with one other figure to its right. In like manner that which is expressed in words by a hundred, is represented in writing by a figure with two others to its right. By this plan it becomes unnecessary to put any mark on a 3 or a 5 to show that it should mean 3 tens or 5 tens of tens, the meaning being shown by the position each figure occupies in a row.

15. The units of the several degrees (11) have their value represented by the place in which they stand, beginning at the right hand, as follows:—

A number meaning	units only is a unit of the	first	place.
....	tens	second	...
....	hundreds	third	...
....	thousands	fourth	...
....	ten-thousands	fifth	...
....	hundred-thousands	sixth	...

Thus, in the line of digits,

6666666

every 6 has a different value: the first one on the right hand means simply 6; the second has one figure on its right, and means 6 tens; the third has two figures after it, and means 6 hundreds; the fourth having three to its right, means 6 thousands; the fifth, 6 tens of thousands; the sixth, 6 hundreds of thousands; and the last, which has 6 figures to its right, means 6 millions. Each of the figures will be seen to mean 10 times more than that on its right. The whole will be read thus: six millions, six hundred and sixty-six thousands, six hundred and sixty-six.

Exercise II.

In the following collections underline all the figures which mean—

I. Tens. 575; 64; 8297; 48623; 59; 161; 3287; 15.

II. Hundreds. 287; 35698; 4523; 889621; 3391; 178; 39625; 84721.

III. Thousands. 2896; 5832145; 627489; 82713; 9862; 59418; 327145; 56732.

IV. Millions. 8274638857; 94728547; 832749683; 7283471.

16. The cipher* (0) has no meaning in itself, but is only useful in determining the place of other figures.

To represent the number four hundred and five in writing two figures only will be needed, one to signify four hundred, and the other, five; but if these two figures are set down together thus, 45, the 4 will be mistaken for 4 tens, being only in the second place. To mean four hundreds, the figure 4 should have two figures to its right (15); the cipher (0) is therefore put in the place usually given to tens to show that the number is composed of hundreds and units only, and that there are no tens. Four hundred and five is therefore written 405. So six thousand and ninety, consisting only of 6 thousands and 9 tens, will be written thus, 6090; the 9 needs the cipher to show that it means nine tens, and not simply nine, and the 6 must have 3 figures to its right (15) in order that it may be seen to mean six thousand, and not more or less. The 0 is therefore only used to keep every number in the place suited to its meaning.

Exercise III.

Write out in words the separate value of every figure in the following expressions:—

Example.—56802; fifty thousand, six thousand, eight hundred, and two, or fifty-six thousand eight hundred and two.

2305; 806; 7095; 20300; 457298; 627421; 33911; 427816; 9032804; 8271096; 32745841; 72918; 13; 5172; 840; 621.

17. *In reading* figures it is usual to break up the whole into periods of three figures each, beginning with the unit.

Observation.—We have a name (tens) for all the units of the second place, another (hundreds) for units of the third place, and one (thousands) for units of the fourth place. But for units of the fifth place we have no new name, but are obliged to combine two of the words already employed—they are called *tens of thousands.* So, also, we have no special name for units of the sixth place, for we call them hundreds of thousands. But for units of the seventh place (thousands of thousands) we use the word millions. Beyond this, numbers are generally called tens, hundreds, or thousands of millions, according to their nearness to the seventh figure. The word hundreds, for example, is used twice when we read a line of

* *Cipher*, from an Arabic word, meaning empty or void.

six figures, and three times when we read a line consisting of nine figures. Thus the figures 763,842,517 are read, seven *hundred* and sixty-three millions, eight *hundred* and forty-two thousands, five *hundred* and seventeen. That is, we speak first of hundreds of millions, secondly of hundreds of thousands, and lastly of simple hundreds (hundreds of units). The last three figures refer to units only, the next three to thousands, and the rest to millions. It is usual, therefore, to mark off the last three figures in the row and consider them as units, the next three as thousands, and the remainder as millions. But this is quite arbitrary, and is merely an arrangement for convenience.

Thus the same set of figures, 7198324, may be read in several different ways. I. Taking one figure at a time, 7,1,9,8,3,2,4 = 7 millions, 1 hundred thousands, 90 thousands, 8 thousands, 3 hundreds, 20, and 4. II. Taking periods of two, 7,19,83,24 = 7 millions, 19 tens of thousands, 83 hundreds, and 24.

EXERCISE IV.

Point off the following lines of numbers, and read them in periods of three:—

17094632; 508704602; 290782; 5069413; 8274169325; 2748629174; 3072; 89621543; 728034; 6195; 83274; 409608; 30729; 8504640; 3270562; 92807; 50963; 827041; 2031.

18. To write down in figures any number which may be described to us in words, it is only necessary to remember that in our system of notation the value of every figure is known by the number of those which stand on its right. If we wish to write 8 hundreds, we have only to take care that the 8 shall stand in the third place from the right, putting ciphers on its right if we have no other numbers to fill those places.

EXERCISE V.

Represent in figures the following expressions:—

Nine hundred and eighty; forty thousand and two; seven thousand six hundred; eighty-one thousand four hundred and two; two hundred and fifty; five millions four thousand and seven; eight thousand six hundred; twenty-four thousand nine hundred and five; twelve hundred and fifteen; six thousand four hundred and ten; nine hundred and eighty-one; eighteen millions and six; four hundred and thirteen; five hundred.

** SECTION II.—VARIOUS METHODS OF NOTATION.

19. The plan of representing large numbers by varying the position of a few figures would apply equally well if any other number than ten were chosen as the basis of our notation.

It has been seen that in our own *decimal* system we require nine significant figures only, and all numbers, however great, are expressed by placing these in certain positions. Similarly, if we had chosen to make the number four the standard of our calculations, we should only have required three digits, the number four would have been represented as 10, five as 11 or four and one, six as 12 or four and two, and sixteen, or four times four, would have been expressed in the same way as we now write ten times ten, viz., 100. So, also, if we had chosen six as the standard number for our notation, the number 235, which we are now accustomed to call 2 tens of tens, 3 tens, and 5 units, would have represented 2 sixes of sixes, 3 sixes, and 5 units. The following table will show how the written characters, to represent the first twenty numbers, might have been varied if the basis of a system of notation had been either two, six, eight, or twelve.

Decimal Scale.	Binary Scale. (base 2.)	Senary Scale. (base 6.)	Octary Scale. (base 8.)	Duodecimal Scale. (base 12.)
1	1	1	1	1
2	10	2	2	2
3	11	3	3	3
4	100	4	4	4
5	101	5	5	5
6	110	10	6	6
7	111	11	7	7
8	1000	12	10	8
9	1001	13	11	9
10	1010	14	12	a
11	1011	15	13	b
12	1100	20	14	10
13	1101	21	15	11
14	1110	22	16	12
15	1111	23	17	13
16	10000	24	20	14
17	10001	25	21	15
18	10010	30	22	16
19	10011	31	23	17
20	10100	32	24	18

20. The figures 1, 2, 3, &c., which are in common use in England, and in most other parts of Europe, are usually called *Arabic.* Their use became familiar among Arabic writers on Mathematics and Astronomy in the tenth century. But they were first employed in Europe by the Arabs, or Moors, who during several centuries occupied Spain, and did much to diffuse throughout Western Europe a love of calculation and of science.

It is generally believed, however, that these symbols were used long before by the Hindoos, and that the Arabs learnt them of that people. They were brought into general use in Italy by Pope Sylvester II., but were not universally adopted throughout Europe till the fifteenth century. In several other nations, the letters of the alphabet have been used to represent numbers; thus the Jews employed the first ten letters of the Hebrew alphabet to stand for the first ten numbers, and the remaining letters to stand for different collections of tens.

21. A similar plan of notation was adopted by the Greeks. Their alphabet contained 24 letters; to these 3 other characters were added, and thus, for the purpose of notation, they possessed 27 characters, or 3 nines. Of these the first nine were used for the nine units (from 1 to 9); the second to represent the nine tens (10 to 90); and the third for the nine hundreds (from 100 to 900). By combining these characters it was very easy to express any number up to 999. Higher numbers were represented by placing a point or dash under any one of them, and thus it was multiplied by 1000. Hence $\beta = 2$, but $\underset{,}{\beta} = 2000$. The letter μ (the initial of *μυρια* or 10,000) placed under any letter was used to multiply it by 10,000, thus $\beta_{\mu} = 20{,}000$. This system possessed many advantages, its chief defect being the absence of a cipher.

22. The earlier notation in use among the Greeks was much more cumbersome. Simple strokes were used for the first four numbers, thus, I, II, III, and IIII, but for five they used the first letter (π) of the word *pente* (πέντη) five; for ten the first letter (Δ) of the word *deka* (Δέκα) ten; for ten times 10 or one hundred, the first letter (H) of the word *hekaton* (Hεκατον); and for a thousand the first letter (χ) of the word *chilia* (χιλια) = a thousand. Five times any number was expressed by putting the letter π over that number, thus $\overline{|\mathrm{III}|}$ meant five times three; $\overline{|\Delta|} = 50$, or 5 times 10; $\overline{|\mathrm{X}|} = 5 \times 1000$, or 5000.*

* We have two familiar examples of the use of an entire alphabet for numbering. The 119th Psalm consists of 22 portions, which are distinguished by the 22 letters of the Hebrew Alphabet. Each of Homer's great poems, the Iliad and Odyssey, consists of 24 books, and these are also distinguished by the 24 Greek letters. But this method of notation was of late date, and was seldom used.

23. The only one of the ancient systems of notation which we ever employ is that which was used by the Romans, and which is still found painted on clock faces, and placed at the heads of chapters in English books. The following table will show the several varieties of form which this singular system included.

1. I.	16. XVI.	400. CCCC.
2. II.	17. XVII.	500. D, IↃ.
3. III.	18. XVIII, XIIX.	600. DC, IↃC.
4. IIII. IV.	19. XVIIII, XIX.	700. DCC, IↃCC.
5. V.	20. XX.	800. DCCC, IↃCCC.
6. VI.	30. XXX.	900. DCCCC, IↃCCCC.
7. VII.	40. XXXX, XL.	1000. CIↃ, M, Ī.
8. VIII, IIX.	50. L.	2000. CIↃCIↃ, IICIC, IIM.
9. VIIII, IX.	60. LX.	5000. IↃↃ, V̄.
10. X.	70. LXX.	10000. CCIↃↃ.
11. XI.	80. LXXX, XXC.	50000. IↃↃↃ, L̄.
12. XII.	90. LXXXX, XC.	100000. CCCIↃↃↃ.
13. XIII, XIIV.	100. C.	500000. IↃↃↃↃ.
14. XIIII, XIV.	200. CC.	1000000. CCCCIↃↃↃↃ.
15. XV.	300. CCC.	

The principle of determining the value of a figure by its local position is only applied in a very limited way here. V stands for five, but IV means one less than five, while VI means one more than five, or six. In like manner, L expresses fifty, and C a hundred, but XL means ten less than fifty; LX ten more than fifty; XC a hundred all but ten; CX a hundred and ten.* That is, *whenever a character representing any number stands to the left of one representing a larger number, the value of the first is meant to be taken from that of the second; but whenever the characters are otherwise arranged, the separate values of each are to be added together.*

24. Much difficulty has been felt in accounting for the choice of the letters V, X, D, M, &c., for the Roman notation. And it has been conjectured that the first nine numbers were represented by simple strokes, thus, |, ||, |||, ||||, &c., and the ten at first by nine strokes with a bar across them (), and afterwards, for convenience, by the simple cross (×). Two strokes being thus required for ten, three were used for a hundred, thus, ⊏ or C, and four for a thousand, thus, M, or as found in old MSS. . Now the half of X is V,† the half of ⊏ is ∟, and the half of  was

* In some cases this method of subtraction is carried further, and IIX has been found used for 8, or 10 − 2, and XXC for 80. This plan of reckoning by defect instead of addition is quite peculiar to the Roman system.

† V was the old form of writing the vowel *u*, which is the *fifth* of the series of vowels. Some writers attribute the use of V for five to this circumstance, but there is no evidence that the coincidence is more than accidental.

originally written IↃ, and is easily contracted into D. It is very probable, however, that the C and M were chosen because they are the initials of the words *centum* a hundred, and *mille* a thousand.

25. In point of practical utility and convenience this system is far inferior to that of the Greeks. It is only fitted to register large numbers, and computation is almost impossible with it. But the Romans paid very little attention to Arithmetic as a science, or indeed to any branch of Mathematics. They only used numbers for business transactions; and their calculations, owing to their defective notation, were very laborious, mechanical, and involved. Slaves were kept in most large establishments for the express purpose of keeping accounts, and they appear to have used either a *loculus*, a box of pebbles, or an *abacus*, a sort of frame with a number of wooden balls which could be arranged in columns to register units, tens, hundreds, &c., &c.

Questions for Examination.

Define arithmetic, number, a unit. What is meant by the arbitrary selection of a unit, and when is it employed? Distinguish between abstract and concrete numbers? What is to be understood by a system of numeration? What by a system of notation? What are the necessary requisites of every system of notation? What truth in arithmetic forms the foundation of all such systems? Give examples.

What is the meaning of the word decimal? How is it applied in numeration? When is a system said to be decimal? To what circumstance is the choice of the decimal method commonly attributed? Describe any possible method of representing numbers decimally which differs from our own.

What is the distinguishing feature of our plan of decimal notation? How are we to make any given number represent hundreds, thousands, millions? If there is a number, 8 for example, how must it be placed so that it shall mean 8 hundred thousand, or 8 tens of millions?

Write down the first twenty numbers as they would be represented if 5 were the basis of our system, instead of 10. Write similar lists in each of the other scales up to 12. How many digits are required in each system? What is the best practical rule for reading off a line of figures readily?

What is the use of the cipher? What is the origin of the figures in ordinary use? Describe the two systems of Greek notation. Show their defects. Write the numbers, 54, 709, 8532, in Roman characters. What was the chief peculiarity of the Roman notation? Account for the use of its characters. Describe some other systems of notation very different from our own. In what respects are they different?*

* As the methods of transferring numbers from the decimal to any other form of expression, require the use of several more advanced rules in arithmetic, the further consideration of this subject, and some exercises upon it, are reserved for the Appendix.

ADDITION AND SUBTRACTION.

SECTION I.—SIMPLE ADDITION.

26. Addition is the process of finding one number which shall be exactly equal to two or more other numbers; or of placing several collections of units together and finding one expression which represents the amount of them all.*

For instance, when we propose this question, "How many will 5 apples and 12 apples and 6 apples make if all put together?" Or, "If 25 sheep are in one field and 14 in another, how many would there be if they were all turned into one?" we ask a question in Addition, for it is only by *adding* or combining the several numbers mentioned that we can obtain the answer we seek.

27. The answer to any question in Addition is called the SUM of the numbers mentioned in the question.

Thus, if four and five make nine, nine is called the *sum* of four and five.

* *Preliminary Mental Exercise.*—It is useful for the student to be practised well in the addition of small numbers before doing sums on a slate. Thus, to 4 add 7 and 7 and 7, &c., or to 5 add 6 and 6 and 6, &c., and so on with each of the nine digits in turn. When all combinations of numbers have been exhausted in this way, the pupil will be able to carry any such series up to a hundred very rapidly. Then miscellaneous numbers may be given, as 9, 7, and 8; 15 and 14; 27 and 19; 8, 6, and 12, &c., until considerable facility has been obtained.

When such exercises have been repeatedly given the student will be prepared to cast up rapidly, without mentioning the figures themselves, but only the result of each addition, any ordinary column of figures, as,

7
9
8
3
2
5
1
6
8
—

It is evident that the sum of these numbers will be the same in whatever order they are placed; hence it is a good plan, after getting an answer to one of these sums, to begin again at the opposite end of the column and add them backwards. Thus, beginning at the foot of the first sum, the learner should not say 8 and 6 are 14, 14 and 1 are 15, &c., but 8, 14, 15, 20, 22, 25, 33, 42, 49. To prove that he is right he may then begin at the top and say 7, 16, 24, 27, 29, 34, 35, 41, 49. The more rapidly he can repeat these numbers in succession, the more likely he will be to obtain the right answer. Slow computers are generally inaccurate.

28. *Signs.* $+$, called *plus*, is the sign of addition. Whenever it is found between two numbers it directs us to add them together, *e.g.*, $7 + 6$ means seven *and* six.

$=$, called *equal*, is the sign of equality. Whenever it is found between two expressions it signifies that the quantity represented on the right amounts to the same as that represented on the left.

Thus $6 + 5 = 11$ (six *plus* five *equals* eleven) means that the six and the five taken together make up the number eleven.

Or, $a + b + c = d$, means that the *sum* of the quantities called a and b and c make up the quantity called d.

EXERCISE VI.

(a). Express each of the following numbers in four different ways, decomposing them into parts which are equal to themselves.

Example.—$25 = 2$ tens $+ 5 = 18 + 7 = 10 + 15 = 16 + 9 = 12 + 6 + 7$, &c.

74; 120; 36: 273; 85; 884; 3796; 24; 83; 127; 29; 65; 128; 230; 472; 191; 654; 37; 86; 94; 72; 230; 719; 685; 32.

(b). Decompose the following numbers into their equivalents so that one of each denomination shall be carried a step lower and added to the next.

Example.—In 7594, take the first figure 7, which means 7000, consider this as 6000 and remove the remaining thousand into the hundreds' place, we have then 15 hundreds; let us call this 1400 and join the remaining hundred to the tens; let the 19 tens thus formed be regarded as $18 + 1$, and unite the 1 ten to the last figure 4: the whole will then stand

$7594 = 6000 + 1400 + 180 + 14.$

So also, $5783 = 4$ thousands $+ 16$ hundreds $+ 17$ tens $+ 13$.

6234; 5987; 4261; 54823; 2096; 83427; 1964; 387; 609541; 83271; 45918; 64729; 803245; 371296.

29. AXIOM II.—*If we add all the parts of any numbers together, we add the numbers themselves.**

Demonstrative Example.—The number 14 may either be added to another as a whole, or ten may be added, and then four; or it may be resolved into $7 + 5 + 2$ and added in that form.

So $7 + 6 + 5 = 5 + 7 + 6 = 5 + \overline{4 + 3} + 6$.

General Formula.—If $a + b = x$ and $c + d + e = y$

then $a + b + c + d + e = x + y$.

30. AXIOM III.—*Concrete numbers can only be added together when they refer to the same objects.*

Demonstrative Example.—5 horses and 7 sheep do not make either 12 horses or 12 sheep. The number 12 cannot properly describe these collections at all, unless we use some word such as quadrupeds or animals, which applies equally to both horses and sheep, and say 5 horses + 7 sheep = 12 quadrupeds. In like manner, £8 and 6s. cannot be said to make 14 of either kind; and the two quantities cannot be added together so as to make one sum of money, unless they can both be first represented in concrete numbers of the same kind, as, 160s. and 6s. together make 166s.

In the same way, whenever any special meaning is attached to any figures, those only can be added together which have the same meaning. In 500, for example, the 5 represents 5 units of the third degree, or 5 hundreds, and in 60 the 6 represents 6 units of the second degree. Now these two numbers cannot properly be put together and said to make 11 of any kind of unit.

* We pursue this plan in all other parts of arithmetic, treating numbers piecemeal, and counting them, part by part, for this reason: *We only know the relations between a few small numbers; these we learn by habit and practice, and it is by considering all larger numbers broken up into such parts as we know, and operating upon these parts successively, that we operate upon the whole.* All arithmetic consists of contrivances for resolving long or complex operations into a series of simple ones. For we have no power which will enable us to say at once how much two hundred and eighty-seven and seven hundred and fifty-six make when added together. But we can first add the two hundred and the seven hundred, then the eight tens and the five tens, and lastly the seven and the six. Each of these operations is familiar to us by itself; but very few people could leap to the answer at one step.

31. Hence it is necessary to arrange our sums so that the eye shall immediately perceive the numbers which ought to be added together. Thus if we have £10 9s. 6d. and £7 4s. 2d. and £3 8s. 7d. to add together, it will be convenient to place the pounds in a column by themselves, and the shillings and pence also in vertical lines, thus :—

£.	s.	d.
10	9	6
7	4	2
3	8	7
20	21	15

we then perceive at once what figures they are which may properly be added, and then it is readily found that all the parts of these several sums may be expressed as £20, 21 shillings, and 15 pence. In like manner, suppose we wish to tell the amount of the following numbers, $794 + 28 + 3057 + 652 + 7$, it will be more convenient to place them in columns also, thus :—

Thous.	Hunds.	Tens.	Units.
	7	9	4
		2	8
3	0	5	7
	6	5	2
			7
3	13	21	28

Those numbers which represent units of the same degree are now in the same vertical lines, and by adding up each column by itself it is found that all the parts of the given numbers taken together produce 3 thousands, 13 hundreds, 21 tens, and 28 units.

But since (15) the value of each separate figure depends on the number of digits to its right, it is not necessary to use lines to separate the various columns of numbers. They will easily be distinguished by placing the units of each number in the right hand column, the tens in the second, and all figures of the same value in the same column.

32. Axiom IV.—*We do not alter the value of any quantity, if what we take from one part we add to another.*

Demonstrative Example.—The answer in (31), viz., 3 thousands, 13 hundreds, 21 tens, and 28 units, is not given in the most convenient shape. The 13 hundreds evidently consist of 1 thousand and 3 hundreds, so that 3 thousands + 13 hundreds = 4 thousands + 3 hundreds. Similarly, the 21 tens consist of 2 hundreds and 1 ten; 3 thousands, 13 hundreds, and 21 tens are the same, therefore, as 4 thousands, 5 hundreds, and 10. But the 28 units = 2 tens and 8 units, therefore the whole number—

$$\left.\begin{array}{c|c|c|c} 3 & 13 & 21 & 28 \\ 3000 & 1000 + 300 & 200 + 10 & 20 + 8 \end{array}\right\} = 4 \mid 5 \mid 3 \mid 8$$

That is to say, we have *carried* from the units place 20 and placec among the tens, as 2; we have carried 20 tens from the secc column and placed it among the hundreds, as 2; we have carried from the hundreds and placed it among the thousands, as 1. also, the answer, £20 21s. 15d., given in (31), £20 + £1 + 1s. 1s. + 3d. = £21 2s. 3d. By thus transferring these parts fr one place in the line to another, no alteration is made in the val of the line itself, and the answer is expressed more concisely by t second method than by the first. This process is called *Carrying*

33. Since (31) the summing up of all the parts successively is t summing up of the whole, it matters not in what order the seve columns or lines of figures are added up. The result must be t same in any case. For example, in the following sum—Add togetl the numbers 4083 + 769 + 8402 + 1237 + 9626 and 8642— first arrange the units in a vertical column, then the tens, afterwa the hundreds and the thousands, thus:—

4	0	8	3
	7	6	9
8	4	0	2
1	2	3	7
9	6	2	6
8	6	4	2
30	25	23	29
32	7	5	9

We may first add up the thousands and fi them to be 30; then the hundreds, which amou to 25; then the tens, which are 23; and, last the 29 units. This answer, however, wants to corrected by bringing the tens of each denomi tion into the column on the left. But if we h begun with the units column this second ope tion might have been avoided, for, on finding t amount to be 29, it would have been evident th 9 only should remain as units, and that the 20 should be transf red to the next column, under the name of 2 tens. By beginni at the thousands, however, we cannot at once tell how many thc sands will appear in the answer, because some of the hundreds m require to be expressed under that name. Hence it is more *cc venient** always to begin at the right hand.

* It is important that the pupil should not mistake rules of mere convenience rules which involve matters of principle. It will, therefore, be well that a few su should be done in the way just described, before the usual method is adopted.

RULE FOR SIMPLE ADDITION.

34. Arrange the figures so that the units of the same value stand exactly in the same column (31). Add up the figures in the first column on the right (33). If the total is no more than 9 it may be set down at once under the units. If it be an exact number of tens, a cipher (0) must be placed under the units column and the tens removed to the next column, but if more, the units only must be set down, and the tens added to the next column. In the same way add up the tens column, carry the hundreds, if any, to the third column, and place the remaining tens only in the second column. Proceed in this way till each column has been added.

EXERCISE VII.

Add together the following collections of numbers :—

1. 123 + 58 + 4094 + 835 + 6294 + 8327 + 5186.
2. 567 + 90 + 48 + 39 + 4728 + 1000 + 6489 + 327 + 4578.
3. 528 + 347 + 269 + 947 + 2586 + 9324 + 82746 + 5372.
4. 23 + 19 + 48 + 61 + 314 + 1000 + 8031.
5. 7 + 19 + 324 + 8 + 160 + 2430 + 29.
6. 12 + 150 + 3987 + 141 + 5 + 67 + 3005 + 498.
7. 13 + 798 + 230 + 647 + 2350 + 826 + 97.
8. 3290 + 574 + 386 + 2074 + 3826 + 5049 + 2786.
9. 325 + 472 + 569 + 8072 + 40961 + 300040 + 713.
10. 58 + 64 + 9721 + 3720 + 5829 + 6874 + 306 + 594.
11. 37045 + 6879 + 3724 + 4562 + 82971 + 37256 + 409.
12. 2304 + 50695 + 28 + 14 + 3972 + 51 + 694 + 2804.
13. 709 + 8304725 + 627 + 81 + 471 + 391 + 2740 + 83.
14. 200 + 371209 + 621 + 30 + 8594327 + 3269 + 942.
15. Add together three millions and forty, two hundred and five, sixteen thousand eight hundred, twelve hundred and fifty-nine, eighty-six, and ten thousand and four.

16. Add together four hundred and seventy-three, five hundred and ten thousand, sixteen millions, five hundred and eighty, thirty-seven thousand, three hundred and seven.

17. Add together four, five, nine, fifty, three hundred and eight, ten thousand, two thousand and nine, forty-eight, sixteen, and twelve.

18. Find the sum of seventy-eight, two hundred and six, four hundred and eighty-three, twelve thousand, three hundred, fifty-four millions, two thousand six hundred and seven.

19. What is the total amount of three hundred and five, seven thousand and ninety, six hundred and forty-seven, eight thousand three hundred and fifty-four, two hundred and eighty-six, four thousand three hundred and eight, seven hundred and forty-five.

20. A man had in his possession several hoards of money. In one place he had 27 gold pieces and 156 silver ones, in another 758 copper coins, 123 of gold, and 287 of silver; in a third, 96 of each; in a fourth, 37 of gold, 29 of silver, and 100 of copper. How many coins of each kind had he, and what was the total number of pieces?

21. The less of two numbers is 527 and their difference 279, what is their sum? If the less be 782 and the difference 156, what is the sum?

22. A corn merchant had four granaries, the first contained 297 quarters of wheat, 563 of barley, and 641 of oats; the second, 8507 quarters of barley, 709 of wheat, and 56 of oats; the third contained 2634 of wheat, 1617 of oats, and 500 of barley; the fourth contained 728 quarters of each. How much of each kind did he possess, and what was the total quantity of grain?

23. How long is it since the year 1491 B.C.? How long since 347 B.C.? since 4004 B.C.? since 2348 B.C.? since 445 B.C.?

24. How many days from March 5th to Dec. 19th, inclusive?

25. If A possesses £537, B £29 more than A, C as much as A and B together, and D £185 more than the sum of the other three; what is their total possession?

26. A postman delivered 628 letters on Monday, 510 on Tuesday, 496 on Wednesday, 837 on Thursday, and as many in the last two days as in the three previous days, how many did he deliver in the week?

METHODS OF PROVING ADDITION.

35. To prove a sum is to verify the answer, and to show that no other could be correct.

The usual method is to vary the form of the sum and to see whether the same answer arises in both cases by different operations. The simplest way in which this may be done is to cast up each line of figures twice, beginning at the foot of the column in one case and at the top of it in the other. It is so exceedingly unlikely that the *same* mistake should be found in both answers, that when they agree we generally assume them to be right. Again, any line may be selected from the sum, and after the answer has been found, the whole may be worked a second time, omitting that line. If when this line is again added the same answer is found, we conclude that it is the true one.

70962	70962
2385	2385
247	247
(50689)	3247
3247	5469
5469	82310
	50689
132999	132999

In the following example it may be seen that the answer is first found in the ordinary way, and afterwards by the omission of one line, which is added by itself. It is usual, for convenience, to cut off the top line for this purpose, but the principle remains the same.

36. A much more efficient and satisfactory method of proof is as follows. Suppose the sum to be

70962
8342
749
1256
39724
82461
203494
23220

and that the answer found in the ordinary way is 203494; cast up the left hand column instead of the right, thus, 7 and 3 and 8 are 18. But 20 stands underneath; set down the difference between 18 and 20, for this 2 must have been brought from the column to the right. Again, 2 and 9 and 1 and 8 are 20, but the 2 tens of thousands brought from the left, and the 3 thousands, make 23; set down 3 underneath, for this is the difference between the 20 thousands in the column and the 23 thousands in the answer. Again, the sum of the hundreds column is 32; take this from 34 and set down the remainder 2 under the 4. Now cast up the tens column alone; they amount to 27. But because the 2 hundreds already set down, and the 9 tens in the answer, make 29

tens, set down the difference between 27 and 29 under the tens. Now because the sum of the units column is found to be exactly 24, or 4 + 2 tens, the answer is proved to be right.

SECTION II.—COMPOUND ADDITION, OR THE ADDITION OF CONCRETE QUANTITIES.

753
827
462
396

37. Suppose in the following sum the figures in the first column on the left, 7, 8, 4, 3, instead of meaning simply hundreds of units, meant pounds, those in the next column shillings, and those in the third column pence, the same rules that have been already stated will apply exactly. But here the value of the figures in the three columns, though dependent as before on local position, is not regulated by tens, hundreds, &c., but by the number of pence in a shilling, and of shillings in a pound. In the top line, for example, the 5 means 12 times more than if it had been one step to the right, and the 7 means 20 times more than if it had stood in the second column. When, therefore, we add together the numbers in the third line (6, 2, 7, and 3) and find them to be 18, we must consider this number of pence, not as 10 and 8, but as 12 and 6, the 12 pence being 1 shilling. This 1 may, therefore, be carried to the place of shillings, and the 6 only set down to pence; similarly, the column of shillings amounts to 23, but because 23 shillings make £1 and 3 shillings, the 3 only must be placed under the shillings, and the 1 transferred to pounds. The answer to this sum is therefore £16 3s. 1d.

38. Whenever sums of money, or quantities of weight, or length, or other magnitudes expressed by concrete numbers, require to be added together, it is only necessary to know the meaning of the several terms employed (pound, yard, ton, mile, &c.), and how much of any one is contained in one of another name. This knowledge may be obtained from the tables in the Appendix, which should be learnt by heart.

RULE FOR COMPOUND ADDITION.

39. Place those numbers which refer to the same quantities in separate columns (31); add the numbers in each column by themselves, beginning with those of

the lowest value (33), and transfer into the next column as many of the less as make one or more of the greater, placing only the remainder under the first; proceed in the same way until the numbers of the highest denomination are reached, when the process is as in simple numbers.

Example.—Add together

tons.	cwt.	qrs.	lbs.	oz.
7	3	2	27	0
5	11	3	19	5
6	2	2	24	9
5	8	3	9	15
2	13	0	5	12
25	37	10	84	41
27	0	1	2	9

Exercise VIII.

Work out the following Addition sums by the help of the tables :—

1. £27 6s. 4½d. + £308 15s. + £529 6s. 8d. + £34 13s. 9d. + £2000 17s. 8d.

2. £506 18s. 3d. + £27 14s. 3½d. + £9684 3s. 7d. + £12 5s. 3¾d. + £869 14s. 2¼d.

3. £274 8s. 6¼d. + £1200 + £50 4s. 9d. + £783 14s. 5¼d. + £1029 16s. 2½d.

4. £574 15s. 4½d. + £292 18s. 1½d. + £279 16s. 3¾d. + £27 2s. 10¾d. + £69 2s. 6d.

5. 127 cwt. 1 qr. 17 lbs. + 24 cwt. 2 qrs. 27 lbs. + 3 tons 17 cwt. 1 qr. 18 lbs. + 5 tons 6 cwt. 3 qrs. 27 lbs.

6. 32 lbs. 5 oz. 8 dwt. 4 grs. + 3 lbs. 5 dwt. 19 grs. + 4 oz. 17 dwt. 18 grs. + 5 lbs. 6 oz. + 3 oz. 14 dwt.

7. 48 lbs. 13 oz. + 3 cwt. 2 qrs. 9 lbs. + 1 cwt. 3 qrs. 8 lbs. 15 oz. 4 drs. + 3 tons 17 cwt. 6 lbs. + 2 qrs. 9 lbs. 11 oz.

8. 17 miles 3 fur. 19 poles + 28 yds. 2 ft. + 10 in. + 4 miles 3 fur. 8 poles + 7 yds. 2 ft. 9 in.

9. 7 acres 3 roods 9 poles + 2 acres 1 rood 19 poles + 27 acres 3 roods 29 poles 18 square yds. + 52 acres 1 rood 27 poles 12 square yds.

10. 21 qrs. 3 bush. 3 pecks + 5 bush. 7 pecks 2 gals. + 2 bush. 3 pecks 3 gals. 3 qts.

11. Of five packages the first weighs 8 lbs. 3 oz.; the second, 2 qrs. 15 lbs.; the third, 17 lbs. 12 oz. 3 drs.; the fourth as much as the first and third together, and the fifth as much as the first and second together. What is the total weight?

12. From half-past 5 P.M. on the 30th of June to 20 min. to 11 A.M. on the 5th of September, how much time elapses?

13. Add together 29 acres 3 roods 13 poles, 100 acres 2 roods 1 pole, 85 acres 1 rood 29 poles, 71 acres 3 roods 17 poles, and 12 acres 2 roods 18 poles.

14. One room contains 18 square yds. 3 square ft. 19 in.; a second, 42 square yds. 8 ft. 11 in.; a third, 29 square yds. 5 ft. 100 in.; a fourth, 25 square yds. 2 ft. 81 in.; and a fifth as great an area as all the rest together. What is the total surface?

15. What is the sum of 65 gals. 3 qts., 28 gals. 2 qts., 44 gals. 1 qt., and 83 gals. 2 qts.?

16. If I travel from A to B in 5 hours 7 mins., B to C in 8 hours 12 mins. 3 secs., from C to D in 12 hours 8 mins. 14 secs., and from D to E in 5 hours 16 mins. 25 secs.; in what time can I perform the whole journey?

17. What is the united length of 6 roads, one measuring 3 leagues 2 miles 7 fur. 30 poles; another, 27 miles 8 yds.; a third, 27 leagues 6 fur. 25 poles; a fourth, 29 miles 3 fur. 18 poles 2 yds.; a fifth, 19 miles 7 fur. 21 poles 2 ft.; and the sixth as much as the first, third, and fifth together?

18. Add together 63 yds. 3 qrs. 1 nail; 74 yds. 2 qrs. 2 nails; 98 yds. 3 qrs. 1 nail; 103 yds. 1 qr. 1 nail; 14 yds. 3 nails; 27 yds. 2 qrs.

19. Find the sum of 3 miles 6 fur. 128 yds. 9 in.; 7 miles 7 fur. 88 yds. 3 in.; 10 miles 3 fur. 25 yds. 6 in.; and 18 miles 5 fur. 205 yds. 11 in.

20. The following sums are owing to a tradesman, £792 10s., £23 18s., £54 6s. 2½d., and £205 10s. 3d. What is the amount?

SECTION III.—SUBTRACTION.

40. Subtraction* is the process of finding how much greater one number is than another. The excess is called the remainder, or the difference.

Observation.—Subtraction is the opposite of Addition. In Addition two numbers are often given, and we are required to find their sum; but in Subtraction, this sum and one of the two numbers are given, and we are required to find the other number.

Example.—"A man has twenty sheep, and seven of them die, how many has he left?" This is a question in Subtraction. It might be expressed in several other ways, *e.g.*, By how many is twenty greater than seven? or, What is the difference between twenty and seven? or, What number of sheep must be added to seven, to make twenty?

Signs. —, called *minus*, is the sign of Subtraction.

Thus: $20 - 7 = 13$. Twenty *minus* seven equals thirteen. That is, twenty with seven taken away from it, leaves thirteen.

41. Axiom V.—*We subtract one number from another when we take each of the parts of the first away from the second, in any order whatever.*

Demonstrative Example.—If from £50 I first take away 7, then 6, and then 10 more, it is clear that I shall have taken away $7 + 6 + 10$, or £23 altogether. Now the same result would have been obtained had I first taken £20 and then £3, or if I had taken the parts of the 23 away from the 5 tens in any other order.

General Formula.—If $a = b + c + d$,

$$\text{then } x - a = x - b - c - d.$$

* *Preliminary Mental Exercise.*—Before any written sums are attempted in Subtraction, some readiness should be acquired in telling the difference between any two numbers.

One of the best forms of this exercise is the selection of some number, as 20, and taking away some small number, as 3, from it, and then 3 again as rapidly as possible, 20, 17, 14, 11, 8, 5, 2; or begin with 100, and take away fours as rapidly as possible. Exercises of this kind should be repeated until the pupil is able to take 100, or any other large number, and from it subtract a series of 5, 6, or 7, in succession, without hesitation and without mistake. Afterwards more difficult questions may be asked, as, What is the difference between 19 and 5, between 20 and 7, 54 and 19?

In setting down a sum in Subtraction, it is usual to place the smaller number (that which has to be taken from the other) underneath the greater. A line is then drawn underneath, and the answer, or the difference, is placed below it, thus:—

8 apples 3 apples — 5 apples	signifies that if 3 apples be taken from 8, the remainder is 5. The 8 in this case is called the minuend,* the 3 the subtrahend, and the 5 the remainder.

42. The simplest form a Subtraction sum can take will be first considered.

Example.—How many must be added to 342 soldiers to make up a body of 587?

Here 342, the subtrahend, is to be placed beneath 587. The several parts of the one may then be taken from the corresponding parts of the other. Thus, in

587 342 — 245	it appears that 200 + 40 + 5 remain, after taking 300 + 40 + 2 away from 500 + 80 + 7. In such a sum as this it matters little in what order the three separate subtractions are effected; we may either take

the hundreds away first, and the rest afterwards, or we may begin with the units.

Exercise IX.

Find the difference between 256 marbles and 132; between 478 and 215; between 3274 and 1162.

How many must be added to 23 to make 67, to 35 to make 89, to 723 to make 975, to 5321 to make 8765?

43. It often occurs, however, that although the whole subtrahend is less than the minuend, yet some of the parts of the one are greater than the corresponding tens or hundreds of the other. There are two methods of solving such problems; *the method of decomposition* and *the method of equal additions.*

* From *minuo*, to diminish or lessen, and *subtraho*, to subtract. Latin participles having the termination *and* or *end* have always a peculiar meaning. Thus minuend, that which *has to be* diminished; subtrahend, that which *has to be* subtracted; multiplicand, that which *has to be* multiplied; dividend, that which *has to be* divided. Many other words, not connected with arithmetic, serve to illustrate the same form; as memorandum, that which *ought to be* remembered.

Method of decomposition. *Example I.*—"There are 179 cattle in one field, and 342 in another, how many more are there in the one than in the other?" Here we set down the 342, and underneath it the 179; but although the whole represented in the lower line can be taken from the whole in the upper, yet the 9 units of the lower cannot be subtracted from the 2 above it, nor can the 7 tens be taken from the 4 tens. Here, then, a previous Exercise (VI. b.) will be of service, for by it we can decompose the minuend into more manageable portions; thus—

$$\begin{array}{rcl} 342 & = & 200 + 13 \text{ tens} + 12 \\ \underline{179} & = & \underline{100 + 7 \text{ tens} + 9} \\ 163 & = & 100 + 6 \text{ tens} + 3 \end{array}$$

We can easily subtract the lower line from the upper, as every part of the one is now less than the corresponding part above it.

Example II.—Take 3185 ounces from 7263 ounces.

$$\begin{array}{lrcl} \text{By Ex. VI.} & 7263 & = & 7000 + 100 + 15 \text{ tens} + 13 \\ \text{By Ex. III.} & 3185 & = & 3000 + 100 + 8 \text{ tens} + 5 \\ \hline & 4078 & = & 4000 + 0 + 7 \text{ tens} + 8 \end{array}$$

Observation.—Here we take 5, not from 3, but from 13, having removed a ten from the 6 tens in the second place; this leaves 8. Then we have to take 8 tens from 5 tens; but this is impossible, so we withdraw 10 tens, or 100, from the 200 in the next place of the minuend. But 8 tens from 15 tens can now be found; they leave 7 tens. 100 has now to be taken from 100, but as the difference here is nothing, we place a cipher to mark that the hundreds place is empty. 3000 taken from 7000 leave 4000, and the answer is thus found to be 4078.

The method just described, however, though dependent on very simple principles, is not found to be very easy in practice. The ordinary process is explained by the following truth.

44. Axiom VI.—*We do not alter the difference between two unequal quantities if we add equal sums to both.*

Demonstrative Example.—Suppose there are two heaps of stones, one of which contains 50 more stones than the other. It is clear that if 20 more stones be placed on the top of each heap, the one has still 50 more in it than the other. We alter the *magnitude* of each heap by equal additions, but we do not alter the difference between them.

General Formula.—If $a - b = x$,

then $(a + c) - (b + c) = x$.

Hence it follows, that if in trying to find the difference between any two numbers, it is convenient to us to add any quantity or quantities to both of them, we are at liberty to do so; for the remainder, which is obtained after making equal additions to the two original quantities, must be the same as would have been obtained before those additions were made.

Thus if we wish to find the difference between 18 and 58, and we choose to add 2 to both before we work the sum, we are at liberty to do so, for the difference between 18 and 58 must be the same as the difference between 18 + 2 and 58 + 2, or between 20 and 60. This is the sort of operation which we employ whenever we work a Subtraction sum, in which any of the parts of the subtrahend are greater than the corresponding parts of the minuend.

45. *Method of equal additions.*—Suppose, for example, we have to find the difference between the numbers 20245 and 17386.

$$\begin{array}{r} 20245 \\ 17386 \\ \hline 2859 \end{array}$$

We begin this sum by trying to take 6 units from 5; this is impossible. Let us add 10 to the 5; 6 from 15 leave 9. But as 10 were added to the minuend, the same must be added to the subtrahend. There are 8 tens in the next place; let us call this 9 tens, and we shall have added ten to both the upper and the lower lines. Next try to take 9 tens from 4 tens: this cannot be done, so add* 10 tens to the 4; 9 tens from 14 tens leave 5 tens, which we set down. But having added 10 tens, or 100, to the upper line, we must do the same to the lower, by making the 300 into 400; 400 from 200 cannot be taken, so we add 10 hundreds to the upper line, and say, 400 from 1200 leave 800; this is then to be set down. But to compensate for the addition of 10 hundreds to the upper line, the same, or 1000, must be added to the lower. The 17000 then becomes 18000, which taken from the 20000 leave 2000, and the answer to the sum is 2859.

* The term "borrowing," often applied to this artifice, is not employed here, because it is a very misleading one, and tends to conceal from the learner the real nature of the process of subtraction. It is evident that the principle involved in the ordinary method of working this rule is, that of equal additions to minuend and subtrahend, and that nothing whatever is either *borrowed* or paid. It would not be altogether inappropriate to apply the term to the several steps in the process of decomposition described in (43), but here it is only a hindrance to the right understanding of the subject.

46. But it must be carefully noticed, that by this method we have not actually taken away 17386 from 20245, but that both quantities have received an addition, first of 10, then of 100, and lastly of 1000, before the subtracting process was finished. The work shown at length is,

$$\left.\begin{array}{l} 20245 + 1000 + 100 + 10 \\ 17386 + 1000 + 100 + 10 \end{array}\right\} \text{ or }$$

$$\underline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$2859$$

Th.	Hun.	Tens.	Units.
20	12	14	15
18	4	9	6
2	8	5	9

The real subtraction effected has not been of 17386 from 20245, but of 17386 + 1110 from 20245 + 1110; that is to say, 18496 has been subtracted from 21355. But according to (44), the difference between these two latter numbers is the same as that between the two original numbers, and the desired answer is obtained, though by an indirect process.

RULE FOR SIMPLE SUBTRACTION.

47. Place the less number under the greater, arranging the digits as in Addition. Begin at the right hand, and subtract the units of each kind from the corresponding number above; set down the differences underneath.

Whenever a figure in the lower line is greater than that above it, add 10 to the upper, then subtract; and in the next place, on the left, take care to add 1 in the lower line, so that the same sum shall have been added to both.*

* In practice, it is well to consider all Subtraction sums as Addition sums in another form; and when calculating, to mention only the number which has to be added to the minuend, and not the subtrahend itself. Thus in the sum

$$\begin{array}{r} 79234 \\ 32567 \\ \hline 46667 \end{array}$$

a good computer will not say, 7 from 14 are 7, 7 from 13 are 6, &c., but merely looking at the upper numbers he will say, 7 *and* 7 (are 14), 7 *and* 6 (are 13), 6 *and* 6 (are 12), 3 *and* 6 (are 9), 3 *and* 4 (are 7). The words in brackets need not be uttered; it is sufficient to look at the upper row of numbers. It is important in all calculations to use as few words as possible.

Exercise X.

(a). Work the following sums by the method of decomposition, and in the last six sums show at length, as in (43), what is the method employed:—

1. Find the difference between 79 and 85; 640 and 27; 293 and 1000; 41 and 506; 8097 and 73; 50962 and 1238; 4096 and 79.

2. Between 87245 and 37; 2739 and 176; 274 and 8571; 39621 and 2874; 5962 and 4173; 8572 and 4961; 300 and 576.

3. 278 − 37; 400 − 59; 6281 − 497; 30000 − 7294; 4321 − 897; 6210 − 891.

(b). Work the following sums by the method of equal additions, and in the last six show, as in (46), what additions have been made to each:—

4. Take four hundred and ninety-three from a thousand; seventy-nine from two hundred and thirty-six; eighty-four from four hundred and fifty; nine hundred and thirty-seven from two thousand and twenty.

5. Take two thousand six hundred and nineteen from forty thousand and twelve; eight hundred and six from a million; two hundred and forty-five from seven thousand three hundred.

6. 70968 − 32975; 3274 − 596; 30962 − 8147; 100000 − 98735; 56934 − 8973; 20369 − 8725; 40336 − 2972; 81572 − 30961; 48371 − 8961; 12345 − 4532.

7. Find the difference between 7209 and 6995; 48372 and 100628; 4718 and 30198; 72386 and 59421; 87243 and 123961; 79632 and 81405; 2071 and 500000; 5098 and 37.

8. The greater of two numbers is 1004, and their difference 49, what is their sum?

9. Take the sum of 798 and 6251 from the sum of 72835 and 6109. Also, 587 + 6403 from 5962 + 8471 + 9274.

10. If from a sum of £92384, I pay £625 to one person, £804 to another, £2096 to a third, and £1527 to a fourth, how much will be left?

11. How long is it since the year 543 A.D.? since 62 A.D.? since 1057 A.D.? since 1666 A.D.? since 185 A.D.?

12. If the sum of two numbers be 968547, and the less be 209682, what is the greater, and what is their difference?

13. If one person was born in 1810, another in 1795, a third in 1839, and a fourth in 1842, how old would each be in 1854?

14. If a man owes £791 to one person, £683 to another, £4627 to another, and £1629 to a fourth; and if to him one person owes £2086, another £56, and another £5905, how do his affairs stand?

15. Add the sum of 9546 and 3285 to their difference.

16. What number added to the sum of 596 and 3024 will give the sum of 5096, 2837, and 2462?

17. One ship contains 7928 pounds of merchandise, a second contains 39254, and a third 20638. What is the difference between the contents of the first and second, the first and third, and the second and third?

18. Add together 17, 19, 53, and 40, and subtract the sum from that of 19, 64, 83, and 106.

19. $(18 + 6 + 209 + 537) - (628 + 31 + 19)$.

20. $(1786 - 9 - 12) - (235 + 672 - 751)$.

METHODS OF PROVING SUBTRACTION.

48. From (40) it appears that the remainder, or answer, added to the subtrahend ought to make up the minuend. The simplest method of verifying any answer is, therefore, to add it to the lower line in the sum. If the result corresponds to the upper, the answer must be right.

Thus,

$$\begin{array}{rl} 5096384 & = \text{minuend.} \\ \underline{2739725} & = \text{subtrahend.} \\ \underline{2356659} & = \text{remainder.} \\ 5096384 & = \text{remainder} + \text{subtrahend.} \end{array}$$

49. The method of proving Addition described in (36), suggests a similar mode of proving Subtraction. Thus,

$$\begin{array}{r} 96214 \\ \underline{72938} \\ \underline{23276} \\ 1110 \end{array}$$

Begin at the left hand, and set down at each step the difference between the answer thus obtained and the written answer. If after carrying on this difference to the upper figure to the right, and proceeding in this way as far as units, we find that all the differences are accounted for, the answer is right.

SECTION IV.—COMPOUND SUBTRACTION, OR THE SUBTRACTION OF CONCRETE NUMBERS.

50. Let it be required to find how much is left in taking £5 9s. 6d. from £8 3s. 2d.; and suppose that the greater sum of money is in the form of 8 sovereigns, 3 shilling-pieces, and 2 penny-pieces, from which we have to pay away £5 9s. 6d.? It is clear that, although we have enough money and to spare, yet we have not enough pence, nor enough shillings, while the money is in its present form. Such a payment will of course be made by getting change for one of the shillings, and turning it into 12 pence; and by changing one of the sovereigns, in like manner, into 20 shillings. Thus,

£.	s.	d.		£.	s.	d.
8	3	2	will take the form of	7	22	14
			From which, if we take away .	5	9	6
			There will remain . . .	2	13	8

51. The method of decomposition described in (43), as applicable to tens and hundreds, will of course be equally suitable to hours and minutes, or to gallons and quarts, or to shillings and pence, or any form of concrete quantity. When the Tables are known, the learner will easily be able to work the following questions.

EXERCISE XI.

Decompose the following sums in the manner of the two examples, *i.e.*, taking a unit of each denomination except the lowest, and placing its equivalent in value among the numbers of the lower name one step to the right.

Example I.—25 tons 16 cwt. 3 qrs. 4 lbs. 2 oz. = 24 tons 35 cwt. 6 qrs. 31 lbs. 18 oz.

Example II.—12 miles 2 fur. 13 poles 2 yds. 2 ft. 7 in. = 11 miles 9 fur. 52 poles 6½ yds. 4 ft. 19 in.

5 years 217 days 17 hours 54 mins.; 27 weeks 3 days 4 hours; tons 3 cwt. 2 qrs. 10 lbs.; 47 lbs. 7 oz. 6 dwt. 5 grs.; 276 qrs. 3 bush. 2 pecks 1 gal.; 18 hours 2 mins. 17 secs.; 11 yds. 2 ft. 8 in.; 2 miles 3 fur. 7 poles; 5 miles 8 yds.

52. The method of equal additions is, however, that by which such sums are usually worked.

Example I.—Take £79 17s. 3d. from £101 3s. 1d.

£.	s.	d.
101	3	1
79	17	3
21	5	10

=

£.	£.	s.	d.
10 tens	1 + 10	3 + 20	1 + 12
7 tens + 1 ten	9 + 1	17 + 1	3
2 tens	1	5	10

£.	s.	d.
100 + 11	23	13
80 + 10	18	3
20 + 1	5	10

We first try to take the pence of the subtrahend from those of the minuend (3 from 1); this cannot be done, so we add 12 pence to the upper line, 3 pence from 13 pence leaves 10 pence. But having added 1 shilling to the upper line, we must also add 1 to the lower, so we next say 18 shillings from 3; but as this is impossible we add £1 or 20 shillings to the upper line, and say 18 from 23 leaves 5; then having increased the upper line by the addition of £1 we must do the same to the lower, and say, not £9 but £10 from 1; add 10 to this 1 and find what is the difference between £10 and £11. Having set this 1 down we add £10 to the lower line, and say 8 tens from 10 tens leave 2 tens. The answer, therefore, is £21 5s. 10d. It will be seen that £10 + £1 + 1s. have been added to both lines.

	miles.	fur.	poles.	yds.	ft.		miles.	fur.	poles.	yds.	ft.
Example II.—	3	5	23	2	2		3	13	63	7½	2
	2	7	24	4	1	=	3	8	25	4	1
	0	5	38	3½	1		0	5	38	3½	1

1 mile 1 fur. 1 pole have been added to both these quantities.

RULE FOR COMPOUND SUBTRACTION.

53. Place the less number under the greater, and arrange them as in Compound Addition. Begin at the right hand, and take away the parts of the less from the corresponding parts of the greater, setting down the differences underneath. If a number in

the subtrahend be greater than the number above it, add to the upper line as many of that quantity as make one of the next higher denomination; then add one of the same denomination to the subtrahend, and proceed in this manner until the whole subtraction is effected.

Exercise XII.

(a). Solve the following questions; the first six by the method of decomposition, and all by the method of equal additions.

(b). State, in the case of the last ten, what quantities have been added to each. (See Example II.)

1. Find the difference between £79 16s. 3d. and £28 7s. 2½d.; between £125 and £16 2s. 9d.; £826 13s. 4d. and £123 7s. 3d.; £246 13s. 11d. and £298 3s. 6d.; £1000 and £297 16s. 3d.; £5287 16s. 5½d. and £479 15s. 10d.; £279 16s. 3d. and £500.

2. £276 13s. 3d. — £49 17s. 8½d.; £4056 10s. — £274 16s 3¼d.; £2741 5s. 6d. — £3 1s. 8d.; £3004 — £298 16s. 3½d.; £19862 14s. 5½d. — £7934 15s. 6¼d.; £100 — £89 17s. 5d.; £72385 10s. 4¾d. — £6985 12s. 7¾d.—£2728 12s. 5¼d.; £750 10s. — £283 17s. 6½d.

3. 49 weeks — 27 weeks 3 days 5 hours; 23 miles 3 fur. 8 poles — 7 miles 3 fur. 4 poles 2 yds.; 7 leagues 2 miles — 9 miles 3 fur. 12 yds.

4. Take from £1500 the following sums—£158 3s. 7¼d., £62 9s. 7½d., and £54 3s. 4d. How much is left?

5. Find the difference between 1 ton and 7 cwt. 3 qrs. 14 lbs.; between 17 lbs. 13 oz. and 2 qrs. 9 lbs.; between 8 cwt. 1 qr. 17 lbs. and 12 cwt. 3 qrs. 5 lbs.; and between 9 lbs. 10 oz. and 4 lbs. 3 oz. 12 drs.

6. By how much does a road 18 miles long exceed one of 12 miles 3 fur. 17 poles?

7. If I owe £79 16s. 4d., £18 3s. 2d., £27 15s. 3½d., and possess £600, while A owes me £15 9s. 6¼d., and B £23 18s. 2¼d., what balance will remain when all my debts have been received and paid.

8. Subtract 72 tons 15 cwt. 3 qrs. from 100 tons 9 cwt. 1 qr. 12 lbs.

9. Subtract 25 acres 3 roods 4 poles from 105 acres.

10. Subtract 3 leagues 2 miles 7 fur. from 7 leagues 1 mile 9 poles.

11. Subtract 56 yds. 2 qrs. 1 nail from 75 yds. 1 qr.

12. Subtract 5 bush. 3 pecks 2 qts. from 7 qrs.

13. If at 7 mins. past 2 on April 3rd, 1827, a man was 36 years 19 days and 5 hours old, when was he born?

14. If I possess £764 2s. 8d., and B has £29 3s. $5\frac{1}{4}$d. less than I, what sum have we together.

** SECTION V.—ADDITION AND SUBTRACTION.*

54. *If there be three numbers, such that the first is as many more or less than the second, as the second is more or less than the third, the sum of the first and third will equal the second added to itself.*

Demonstrative Example.—12, 7, 2. Because 7 is as much less than 12 as it is more than 2 $\therefore 12 + 2 = 7 + 7$, for, by (32), if what is taken away from one of the sevens be added to another the sum will remain the same.

General Formula.—a, b, c. If b be as many less than a, as it is more than c; or, if b be as many more than a as it is less than c, then $a + c = b + b$.

Exercise XIII.

(a). Find two numbers whose sum will equal $8 + 8$; $12 + 12$; $53 + 53$; $19 + 19$; $100 + 100$; $74 + 74$; $327 + 327$; $568 + 568$; $473 + 473$; $29 + 29$; $58 + 58$; $413 + 413$.

(b). Place in each parenthesis a third number which, if added to the first, will equal the second added to itself;

27, 18, (   ); 47, 54, (   );
3, 17, (   ); 29, 16, (   );
18, 100, (   ); 293, 500, (   );
409, 400, (   ); x, x, -3 (   );
$a - 3$, a, (   ); $a + 5$, a, (   ).

Corollary I.—In any series of numbers so arranged that there is the same difference between every pair of adjacent numbers, the sum of any two which are equally distant from a given number equals that number added to itself.

* The principles laid down in this section are not absolutely necessary to the solution of ordinary questions in the early rules, but they serve to elucidate further the theory of Addition and Subtraction, and to explain several important rules which will hereafter occur. They are placed here because they belong to this subject alone, and are not dependent on any subsequent rules.

Example.—2 5 8 11 14 17 20 23 26

5, for example, is as many less than 14 as 23 is more than 14, hence $5 + 23 = 14 + 14$. In like manner, any number in the series added to itself will give the same amount as the sum of any two which are equally distant from it, as thus, $5 + 11 = 8 + 8$, $17 + 23 = 20 + 20$, $23 + 11 = 17 + 17$, &c.

*Corollary II.**—In every such series of numbers the sum of the two extreme terms equals the sum of any two terms at equal distances from the extremities, and the whole series is made up of a number of pairs, each pair having the same sum as the two extremes.

Exercise XIV.

Make 12 series of 15 numbers each, and show how each illustrates the principle.

55. *If there be four numbers, of which the second is as many more or less than the first as the third is less or more than the fourth, the sum of the first and fourth will equal the second and third.*

Demonstrative Example.—5, 9, 100, 104. For, by (32), if the two numbers, 9 and 100, are taken, and if what is removed from the 9 to make 5 be carried to the 100 to make 104, the sum of the two numbers will be the same; or, $9 + 100 = (9 - 4) + (100 + 4)$; or, $5 + 104 = 9 + 100$.

General Formula.—a, b, c, d. If a be as many more than b, as d is less than c; or, if b be as many more than a, as c is less than d, then $a + d = b + c$.

Exercise XV.

Place a fourth number in each of the following series, so that if added to the first it will equal the sum of the second and third.

9, 7, 20, ( ); 20, 15, 8, ( ); 3, 12, 70, ( ); 29, 5, 64, ( ).
$a + 3$, a, b, ( ); $x - 5$, x, y, ( ); $a + 7$, a, x, ( ).
$m + x$, m, n, ( ).

Suppose $p = q - 6$;
q, p, 15 ( ); p, q, 3 ( ); p, q, m, ( ).

* This corollary forms the basis of an important rule in arithmetical progression and is referred to in the chapter on that subject.

Questions on Addition and Subtraction.

What is Addition? Give an example. Quote one of the principles on which the rule for working it depends. What part of the rule is shown to be necessary by the principle you mention? What sort of numbers are they which ought not to be added together? Why? Repeat the axiom on this subject. What part of the rule does this explain?

Why are numbers placed in columns for Addition? What is carrying? What axiom is assumed when we carry a figure to the next column? What is it to prove a sum? How will you prove an Addition sum? State the Rule for Simple Addition. For Compound. Explain the terms subtrahend, difference, sum, minuend, &c. Describe the method of equal additions. Of decomposition. State each of the axioms assumed in Subtraction, and give an example of each.

Why is it easier to commence a sum on the right hand? What equal additions are always made in a Subtraction sum? Why? How may a Subtraction sum be proved? Give the reason. Give the Rule for Compound Subtraction.

State in words what general truth of Arithmetic is illustrated by each of the following equations:—

1. If $w = x + y + z$, then $a + w = a + x + y + z$
2. and $a - w = a - x - y - z$
3. $a - c = \overline{a + x} - \overline{c + x}$
4. In 20, 25, 30; $20 + 30 = 25 + 25$
5. $(x - y) + (x + y) = x + x$
6. $(x - y) + (a + y) = x + a$

MULTIPLICATION AND DIVISION.

SECTION I.—SIMPLE MULTIPLICATION.

56. To multiply a number is to repeat it, or add it to itself, a certain number of times.

The number thus repeated is called the *multiplicand;** the number of repetitions is called the *multiplier*, and is always an *abstract* number; and the answer is called the *product.*

Observation.—In certain cases it is convenient to call both multiplicand and multiplier *factors*, they being the two numbers which *make* (*facio*) or produce the answer.

57. To multiply is, therefore, to find a number which is as many times greater than the multiplicand as the multiplier is greater than unity.

58. *Signs.* The sign of Multiplication is $\times$, thus, $4 \times 6 = 4$ *times* 6, or 4 *into* $6 = 24$.

So $a \times b = a$ multiplied by b, or taken b times. This is sometimes written ab.

Example.—There are 5 bags of money and 7 sovereigns in each, how many are there in all? Here 7 sovereigns require to be multiplied or repeated, and are called the multiplicand; the figure 5 shows how many times the 7 sovereigns are to be taken, and is called the multiplier; and the answer 35 sovereigns is the product, which is an amount as many times greater than the multiplicand (7) as the multiplier (5) is more than unity (1).

* The word Multiplication literally means manifolding; it is derived from *multus*, many, and *plico*, to fold; duplication, complication, &c., are from the same root. Multipli*cand* means that number which *is to be* multiplied. Multiplier = that which shows how many times this multiplicand is to be taken; and product (from *pro-duco*, to bring forth, means that result which is produced by the process.

Preliminary Mental Exercise.—As in Addition and Subtraction, it is necessary here to make ourselves familiar with the relations between a few simple numbers, in order that we may deal with all larger numbers part by part. For this purpose it is very desirable that the pupil should first make for himself a table, like that in the text, showing the effect of a series of additions to each of the first 12 numbers, and afterwards learn it by heart.

59. In the following Table it will be seen that the first 12 numbers are arranged in a horizontal line at the top, beginning with the number 1. In the second line the number 2 is taken, and a series of twos are added to it in order, so that three twos stand under the number 3, and five twos under the number 5, &c. In like manner a set of equal additions is made to each of the first twelve numbers in successive horizontal lines. Hence, to find 9 times 6, I look along the sixth horizontal line, until I find the number exactly under the 9, which is 54; that is the sum of 9 additions of 6. The product of any other two numbers between 1 and 12 can be found in the same way.

MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

The numbers which form the diagonal of this figure, and which are enclosed in darker lines, are called the *squares* of the numbers to which they belong. It will be observed that the numbers on each side of this line are the same, but in a different order.

60. Multiplication is abbreviated Addition, but it will only enable us to perform those Addition sums in which it is required to add together the *same* numbers.

Example.—If it be required to find what 4 times 57 will make, the sum may be stated as in the margin. In the units column, all the numbers to be added together are alike; we are, therefore, spared the trouble of making 3 different additions to the 7, if we have learnt by heart that 4 sevens make 28; and, in like manner, we need not make several mental efforts in order to add the 5 to itself four times, if we have learnt from the tables that 4 fives make 20. But it is clear, that if all the lines of figures in the sum had not been alike, the Multiplication Table would not have helped us, but we should have been obliged to obtain the answer by ordinary addition.

$$\begin{array}{r} 57 \\ 57 \\ 57 \\ 57 \\ \hline 228 \end{array}$$

61. Multiplication, like Addition and Subtraction, is a distributive operation, *i.e.*, if it is performed on each of the parts separately, it is performed upon the whole. The following axiom is assumed throughout this Rule:—

62. Axiom VII.—*If we multiply all the parts of one number by another, and add the several products together, we multiply the whole of the first number by the second.*

Demonstrative Example.—To multiply 7 by 5, we may take the parts of 7 and multiply each by 5, thus:

Because $7 = 4 + 2 + 1$, then $5 \times 7 = \overline{5 \times 4} + \overline{5 \times 2} + \overline{5 \times 1}$, or $20 + 10 + 5 = 35$.

General Formula.—If $a = b + c + d$,

$$\text{then } na = nb + nc + nd.$$

In multiplying a number, such as 7962, which is composed of four parts, viz., $7000 + 900 + 60 + 2$, by any number (8), it is therefore allowable to multiply each of these parts in order, thus; and the answer thus obtained, 56 thousands + 72 hundreds + 48 tens + 16, is the true one, and only needs to be corrected, as to its form, by the Rule described in (32). Hence,

7	9	6	2
			8
56	72	48	16

63. Case I.—To multiply by any number less than 13—

RULE.

Place the multiplier under the unit of the multiplicand; find the product of the multiplier and number above it; set down the units digit of the answer, carrying the tens into the next place, as in Addition. Then find the product of the multiplier and the figure in the tens place; set down the odd tens, carry the hundreds, and proceed with each figure in the same manner.

Exercise XVI.

1. Find the product of the following numbers, 283 and 7; 50629 and 5; 2964 and 4; 3274 and 8; 59680 and 7.
2. 20847 and 9; 618 and 11; 2934107 and 12; 51839046 and 8.
3. 729 × 6; 84237 × 7; 5864 × 9; 712348 × 9; 58574 × 2.
4. 526387 × 7; 201306 × 5; 4158692 × 7; 723486 × 7.
5. 424374 × 8; 386 × 9; 84729 × 6; 3964187 × 5.
6. 413826 × 6; 702987 × 9; 546208 × 8; 312628 × 7.

64. Axiom VIII.—*Multiplying one number by a second gives the same result as multiplying the second by the first.**

* Advanced students will be interested to know that this principle, which appears almost self-evident, and is assumed without demonstration in ordinary Arithmetic, is made the subject of one of the propositions in the seventh book of Euclid's Elements. The following is the method of proof:—

Prop. XVI. If one number, A, be multiplied by B, the product is the same as if B were multiplied by A; *i.e.*, AB = BA.

Suppose B × A gives a certain result, x, then, according to the definition of Multiplication, x contains B as many times as A contains 1.

$$\therefore \text{(I.)}\ 1 : A :: B : x.$$

Similarly, if A × B gives a certain result, y, then, for the same reason,

$$\text{(II.)}\ 1 : B :: A : y.$$

But by a former proposition, "If the first be to the second as the third is to the fourth, then the first is to the third as the second is to the fourth;" hence the last proposition may take the form, $1 : A :: B : y$, but by (I.) $1 : A :: B : x$. Hence, B is contained as often in x as in y, and $x = y$, or BA = AB.—*Q. E. D.*

Demonstrative Example.—3 times 8 is the same as 8 times 3; for if a number of characters be arranged thus, the whole may either be considered as three rows of eight each, or as 8 rows of 3 each.

0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0

By reference to the Multiplication Table it will be seen that the number 24 occurs twice, once in the third column, on the line of eights (3 times 8), and once in the eighth column, in the series of threes (8 times 3).

General Formula.—$a \times b = b \times a$.

65. *Corollary I.—The product of any number of factors is the same in whatever order they are multiplied.*

Example.—3 times 7 times 6 $(3 \times 7 \times 6) = 6 \times 7 \times 3$, or $7 \times 6 \times 3$.

General Formula.—$a \times b \times c \times d = b \times d \times c \times a$, or $d \times b \times a \times c$.

Observation.—The answer in all such cases is called the *continued product* of the factors.

66. *Corollary II.—We multiply by the whole of a number when we multiply successively by its factors.*

Example.—Because $3 \times 5 = 15$; $7 \times 3 \times 5 = 7 \times 15$. Hence, multiplication by 15 may be effected if we first multiply by 3, and then multiply that product by 5.

General Formula.—If $a \times b = x$, then $nx = (na)\, b$.

67. CASE II.—WHEN THE MULTIPLIER IS GREATER THAN 10, AND ITS FACTORS ARE KNOWN—

RULE.

Multiply by one factor, and that product by another factor, and so on till all the factors have been employed.

Example.—Multiply 3247 by 56.

$56 = 7 \times 8$.

$$\begin{array}{r l}
3247 & \\
\underline{\quad 7} & \\
22729 & = 3247 \times 7 \\
\underline{\quad 8} & \\
181832 & = 3247 \times 7 \times 8 = 3247 \times 56.
\end{array}$$

EXERCISE XVII.

1. Multiply 23972 by 28; 564 by 18; 2740 by 72; 39648 by 56.
2. 20974 by 144; 87412 by 35; 4693 by 49; 847209 by 63.
3. 962187 by 24; 30962 by 55; 8291 by 45; 357 by 240.

68. It follows, from the principle of our notation (16), that to place a cipher (0) on the right of a line of figures has the effect of multiplying the whole by 10. For each figure on being removed one place to the left means 10 times more than it did before. In the same way, placing two ciphers (00) to the right of a number multiplies it by 100; three ciphers, by 1000, and so on with all the higher denominations.

Example.—496 × 10 = 4960. The effect of placing a cipher after the number is to alter the 6 into 6 tens; the 9 which meant 9 tens, into 9 tens of tens, or 9 hundreds; and the 4 which represented 4 hundreds, into 10 times 400 or 4000. Two ciphers would have had the effect of multiplying by 100; three, of multiplying by 1000, &c.

69. CASE III.—WHENEVER ANY NUMBER OF CIPHERS IS TO THE RIGHT OF A MULTIPLIER—

RULE.

> Multiply only by the remaining figures, and add as many ciphers to the answer as there are in the multiplier.

Example.—Multiply 79834 by 400.

First multiply it by 4.

$$\begin{array}{r} 79834 \\ 4 \\ \hline 319336 \end{array} = 4 \times 79834$$

Add 2 ciphers. 31933600 = 4 × 100 × 79834.

70. *Observation.*—This answer has been gained by two steps. Because 400 = 4 × 100, we first multiplied by 4; but in adding two ciphers to this product, every figure in it is made to mean 100 times what it meant before, and the whole has therefore been increased by 4 × 100, or 400 times.

EXERCISE XVIII.

1. Multiply 8746 by 30; 2974 by 200; 8274 by 3000.
2. 56847 by 700; 3178 by 7000; 296845 by 90000.
3. 21869 by 80000; 314721 by 50000; 23896 by 60000.
4. 58471 by 400; 9874 by 80000.

We have now seen how to multiply by any number greater than 10, when that number is capable of being resolved either into simple factors or into 2 factors, one of which is 10, 100, 10000 or one of that series. But there are many numbers, such, for example, as 37, 146, &c., which cannot be treated by either of these methods. The ordinary process of working such questions depends on the following truth.

71. *If the multiplier be broken up into parts the sum of the products of each of these parts and the multiplicand will equal the product of the whole.*

Demonstrative Example.—Because $7 = 3 + 4$, therefore $29 \times 7 = (29 \times 3) + (29 \times 4)$. So if it be required to find the product of 238 and 42, we may take 238×40 and add it to 238×2, and the sum of these products will be the answer.

General Formula.—If $a = b + c$, $xa = xb + xc$.

If this principle be considered in connexion with the preceding (62) it will be seen that both multiplicand and multiplier may be decomposed into parts, and that the following proposition may therefore be assumed.

72. *Multiplication is always effected between two factors, if each of the parts of the one is multiplied by each of the parts of the other, and the sum of their products is taken.*

Demonstrative Example.—25×78.

Here $25 = 20 + 5$, and $78 = 70 + 8$.

then $(20 \times 70) + (5 \times 70) + (20 \times 8) + (5 + 8) = 25 \times 78$.

General Formula.—If $a = b + c$, and $x = y + z$,

then $ax = by + bz + cy + cz$.

73. Let it be required to multiply 723 by 364. Here the multiplier consists of three parts, 300, 60, and 4; but (72) if 723 is multiplied by each of these parts the sum of the products will give

the required answer. We first multiply the parts of the multiplicand (62) by 4, and find the answer to be 2892; we have next to multiply by 60, but since $60 = 6 \times 10$, we may, by (69), multiply by 6 and place a cipher after it, therefore 723×60 is found to be 43380. It now remains to multiply 723 by 300. By (69) this may be done by multiplying by 3 and placing 2 ciphers (00) to the right of the product, hence $723 \times 300 = 216900$. The sum of these three products is (72) the product of 723 and 364, *i.e.*, $(\overline{700 + 20 + 3}) \times (\overline{300 + 60 + 4})$ has been found by multiplying each of the parts of the one factor with each of the parts of the other. The ciphers to the right of the second and third line are unnecessary, the value of the figures being shown by the place in which they stand.

$$
\begin{array}{r l}
723 & \\
364 & \\
\hline
2892 & = 723 \times 4 \\
43380 & = 723 \times 6 \times 10 \\
216900 & = 723 \times 3 \times 100 \\
\hline
263172 & = 723 \times (4 + 60 + 300)
\end{array}
$$

From these considerations we deduce the following rule, applicable to all cases in which the multiplier is a number greater than 10, and whose factors cannot easily be obtained.

RULE FOR LONG MULTIPLICATION.

74. Write the multiplier under the multiplicand. Multiply first by the unit, setting down the answer as in the simple rule; next multiply by the tens, placing the unit of this answer in the tens place. Place, in like manner, the first figure of the hundreds product in the hundreds place, of the thousands product in the thousands place, &c., and add them up, as they stand, to find the total product.

75. Sometimes one or more ciphers occur in a multiplier. Whenever this happens the student has only to remember that units multiplied by units give units as their product; by tens, give tens; by hundreds, give hundreds, &c. It is necessary to think, then, of the value of each portion of the multiplier as it is used, to remember

that the first result obtained has the same value, and to place it accordingly.

Examples.—Multiply 2798 by 204, and 63157 by 3005.

I.

$$\begin{array}{ll}
2798 & \\
\underline{204} & \\
11192 & = 2798 \times 4 \\
\underline{5596} & = 2798 \times 200 \\
570792 & = 2798 \times (200 + 4)
\end{array}$$

II.

$$\begin{array}{ll}
63157 & \\
\underline{3005} & \\
315785 & = 63157 \times 5 \\
\underline{189471} & = 63157 \times 3000 \\
189786785 & = 63157 \times (3000 + 5)
\end{array}$$

76. In (I.) it was necessary to observe that the 2 in the multiplier meant 2 hundreds, and that the first product (twice 8) therefore meant 16 hundreds, and must stand in the hundreds place. In (II.) the 3 of the multiplier meaning 3 thousands, 3 times 7 gives, as a product, 21 thousands, and the 1 must therefore stand in the thousands place, or three figures to the left of the unit.

77. *Observation.*—Although in the process of Multiplication it is easy to see the necessity for beginning the operation on the right, in order that numbers of a certain degree may the more conveniently be transferred from the lower place, there is no reason why the order of the partial multiplications should not be inverted or altered in any way we please. For example:—

$$\begin{array}{ll}
37286 & \\
\underline{264} & \\
74572 & = 37286 \times 200 \\
223716 & = 37286 \times 60 \\
\underline{149144} & = 37286 \times 4 \\
9843504 & = 37286 \times 264
\end{array}$$

Here the first partial product obtained was that of the multiplicand and 200, the unit was therefore put in the hundreds place; the second part of the multiplier was 60, and the first figure of the product took the tens place. The third product consists of a certain number of units, and stands in the same relation to the rest of the sum as if it had been placed first in the usual way.

Exercise XIX.

1. Find the product of 279 and 14; of 8793 and 401; of 17986 and 3709; of 274, 302, and 87; of 53 and 6948; of 77 and 810325; of 968 and 324; of 175 and 809625.

2. Multiply 796284 by 37; 829741 by 59; 304156 by 168; 7428327 by 539; 920685 by 7098.

3. 27468 by 3974; 814729 by 257; 638041 by 724; 57412 by 387; 6541 by 729; 858604 by 23.

4. 5684 × 301 × 29; 407 × 18 × 5; 5096 × 15 × 13; 70271 × 21 × 8; 52965 × 4 × 17.

5. 378596 × 4 × 13; 729 × 8 × 61; 3190865 × 5 × 7; 418327 × 6 × 9; 7184 × 6 × 2.

6. If I have 8 purses containing £17 each; 5 containing £29 each; and 12 £5 notes, how much money have I?

7. What is the difference between the product of 18, 19, and 35, and that of 24, 17, and 12.

8. Find the product of two numbers, the greater of which is 1694, and the difference 189.

SECTION II.—COMPOUND MULTIPLICATION, OR THE MULTIPLICATION OF CONCRETE QUANTITIES.

78. Compound Multiplication is effected by multiplying the several quantities one by one, and placing the answers as in Addition.

The method of Long Multiplication employed in (74) depends on the fact that each figure of the multiplicand, if removed one place to the left, would mean ten times as much. It is therefore not suitable for lines consisting of pounds, shillings, and pence, or any other concrete numbers. For example: the number 283 would be multiplied by 10 if a cipher were added. It would then be 2830. But 2 miles 8 fur. 3 poles could not be treated in this way, but would require to be separately multiplied. The other method of Long Multiplication is, therefore, to be used for concrete numbers.

RULE.

79. When the multiplier is not greater than 12 begin with the numbers of the lowest name, multiply each in turn, and carry the remainder as in Addition.

But when the multiplier is greater than 12 resolve it into its factors, and multiply by each in succession. If no factors can be found which will exactly produce the multiplier, take the nearest, then multiply the top line by the difference between this product and the multiplier, and add this product to the rest.

Example I.—Multiply £63 17s. 10½d. by 274. Here we take 270 and observe that it is made up thus: $9 \times 3 \times 10 = 270$. The sum of money must therefore be multiplied first by these numbers in succession, and afterwards 4 times the original sum should be added to the product.

£.	s.	d.	
63	17	10½	$\times$ 274 $(\overline{9 \times 3 \times 10} + 4)$
		9	
575	0	10½	= 63 17 10½ $\times$ 9
		3	
1725	2	7½	= 63 17 10½ $\times$ 9 $\times$ 3 (27)
		10	
17251	6	3	= 63 17 10½ $\times$ 9 $\times$ 3 $\times$ 10 (270)
255	11	6	= 63 17 10½ $\times$ 4
17506	17	9	= 63 17 10½ $\times$ 274

Example II.—Multiply 17 cwt. 3 qrs. 20 lbs. 7 oz. by 97.

tons	cwt.	qrs.	lbs.	oz.	
0	17	3	20	7	$\times$ 97 $(\overline{12 \times 8} + 1)$
				12	
10	5	0	21	4	= 17 cwt. 3 qrs. 20 lbs. 7 oz. $\times$ 12
				8	
82	1	2	4	0	= 17 cwt. 3 qrs. 20 lbs. 7 oz. $\times$ 12 $\times$ 8 (96)
1	17	3	20	7	
83	19	1	24	7	= 17 cwt. 3 qrs. 20 lbs. 7 oz. $\times$ $(\overline{12 \times 8} + 1)$ 97

EXERCISE XX.

(a). Work the following sums in Compound Multiplication.

(b). Set down, in the first six, the separate value of each line, as in the examples just given.

1. Multiply £7 3s. 8d. by 5; £14 15s. 6d. by 4; £2 9s. 3¾d. by 7.
2. £17 8s. 3d. by 11; £34 17s. 8½d. by 9; £235 6s. 3¼d. by 6.
3. £2745 18s. 11½d. by 8; £695 14s. 2½d. by 15; £27340 18s. 3d. by 24.
4. £5629 7s. 2½d. by 95; £37205 by 78; £46027 4s. 2¼d. by 73.

5. £8954 7s. 4¼d. by 179; £7219 12s. 5d. by 387.

6. 13 acres 3 roods 25 poles by 7; 4 tons 17 cwt. 19 lbs. by 8; 4 miles 7 fur. 8 poles by 13.

7. 17 miles 7 fur. 29 poles 3 yds. by 27; 17 lbs. 10 oz. 15 dwt. 4 grs. by 19; 5 lbs. 7 oz. 2 scruples by 46.

8. 3 cwt. 2 qrs. 7 lbs. × 29; 7 tons 2 cwt. 18 lbs. × 35; 14 cwt. 2 qrs. 27 lbs. × 138.

9. 17 dwt. 4 grs. × 18; 8 oz. 9 dwt. 12 grs. × 41; 7 lbs. 11 oz. 19 grs. × 58.

10. 15 gals. 2 qts. 1 pint × 8; 9 gals. 3 qts. × 18.

11. 35 acres 2 roods 19 poles × 68; 27 acres 19 poles × 26; 7 acres 3 roods 13 poles × 58.

12. 27 leagues 2 miles 6 fur. 2 yds. × 17; 7 miles 2 fur. 28 poles × 59.

13. 5 yds. 3 qrs. 1 nail × 15; 43 yds. 1 qr. 2 nails × 53.

14. 23 weeks 5 days 7 hours × 9; 41 weeks 3 days 27 mins. × 42.

15. Multiply 76 qrs. 4 bush. 3 pecks separately by 18, 52, and 21.

16. Multiply £29 18s. 6½d. separately by 14, 29, and 108.

17. Multiply 372 acres 3 roods 29 poles separately by 26, 28, and 54.

18. Find the cost of 117 lbs. at £2 6s. 10½d. per pound.

19. Find the cost of 2300 gallons of spirits at 12s. 8d. per gallon.

20. If an acre of land produce 37 bush. 23 qts., what will 127 acres produce.

SECTION III.—ABBREVIATED METHODS OF MULTIPLICATION.

80. The following methods will often be found of service in shortening the operations of this rule.

CASE I.—WHEN THE MULTIPLIER OR MULTIPLICAND, OR BOTH, HAVE ANY NUMBER OF CIPHERS TO THE RIGHT—

RULE.

Multiply the significant figures only, and add to the product as many ciphers as there are in both factors.

Example.—27000 × 89600.

Here the 7 of the multiplicand means 7000, and the 6 of the multiplier means 600; therefore the one may be considered as 27 × 1000, and the other as 896 × 100. But by (65) the factors of these numbers may be multiplied in any order, and the product of the 100 and 1000 being 100,000, there remains the product of 27 and 896 only to find; five ciphers added to this product will give the desired answer.

Exercise XXI.

1. Multiply 26280 by 45000; 3729000 by 89680000; 4520 by 3300.

2. 14700 by 170; 10740 by 36000; 5209000 by 170.

3. 2874600 by 3970; 52860 by 2900.

81. Case II.—When the multiplier contains the figure 1—

Rule.

> Add in the figures of the multiplicand at each of those steps of the first partial product under which the same figures would have been placed.

Example.—Multiply 279386 by 18.

Ordinary method.

```
 279386
     18
-------
2235088
279386
-------
5028948
```

But as in this case each figure of the top line has had to be added in order, to that figure of the product which stands one place to its left, it would be easier to make this addition at once, instead of waiting to set it down underneath after the first product had been obtained. The sum would then have been worked thus: 8 sixes are 48, set down 8 and carry 4; 8 times 8 are 64 and 4 are 68 *and* 6 are 74, taking in the last figure of the multiplicand; then 8 × 3 = 24, add 7 *and* 8, the answer is 39. In this way the whole is effected in one line and by a shorter process. This, of course, applies equally when the 1 is in the hundreds or thousands place.

```
 279386
     18
-------
5028948
```

Exercise XXII.

1. Work the following sums in one line—27896 × 17; 30584 × 13; 5196 × 14.

2. 48321 × 15; 70968 × 14; 5096384 × 19; 4078 × 17.

3. 419635 × 16; 210478 × 17; 3121076 × 18; 51943 × 18.

4. 4096 × 19; 123691 × 15; 51427 × 17; 31265 × 18.

5. Work the following in two lines—5081 × 217; 50693 × 189; 71546381 × 231.

6. 718346 × 179; 31456 × 127; 549607 × 518.

7. 21096 × 371; 85462 × 415; 430695 × 163.

8. Work the following in three lines—41863 × 2187; 145963 × 8194; 716384 × 5172.

9. 814065 × 2134; 30796 × 8143; 51968 × 3147.

10. 783260 × 9168; 4718 × 53729; 39682 × 4319.

82. CASE III.—WHEN THE MULTIPLIER IS NINE OR CONSISTS OF ANY NUMBER OF NINES—

RULE.

Add as many ciphers to the multiplicand as there are nines in the multiplier, and subtract the multiplicand from the result.

Example.—Multiply 32682 by 999.

Here, because $999 = 1000 - 1$, we may take the multiplicand a thousand times, and then subtract it once, and the required answer will be obtained. But to multiply by 1000 (68) we have only to add 3 ciphers. Hence,

$$\begin{aligned} 32682000 &= 32682 \times 1000 \\ \underline{32682} &= 32682 \times 1 \\ 32649318 &= 32682 \times (1000 - 1) = 32682 \times 999 \end{aligned}$$

Observation.—A similar principle applies to cases in which the multiplier *exceeds* any power of ten, by one. Here the ciphers will be placed as before and the multiplicand *added*. But when the truth of the former method is understood, this will be too simple to require explanation.

EXERCISE XXIII.

Find the following products—

1. 8096 × 99; 54032 × 999; 4865 × 9.
2. 7038 × 101; 799263 × 8001; 171836 × 9999.
3. 54185 × 10001; 7218 × 1001.
4. 21386 × 11; 50968 × 99; 37254 × 99999.

83. CASE IV.—WHEN ANY ONE DIGIT OF THE MULTIPLIER IS CONTAINED IN ANOTHER PART OF IT—

RULE.

Multiply first by the less, and afterwards multiply this product by the number of times the greater contains the less. Then add up as usual.

Examples.—Multiply 8427 by 364; also 51240 by 742.

I.

$$\begin{array}{rl} 8427 & \\ 364 & \\ \hline 33708 & = 8427 \times 4 \\ 303372 & = 8427 \times (90 \times 4) \\ \hline 3067428 & = 8427 \times 364 \end{array}$$

II.

$$\begin{array}{rl} 51240 & \\ 742 & \\ \hline 35868000 & = 51240 \times 700 \\ 2142080 & = 51240 \times (6 \times 7) \\ \hline 38010080 & = 51240 \times 742 \end{array}$$

In (I.) the first line of the product was multiplied by 9 in order to give the second line, because $36 = 9 \times 4$. In (II.) the first line of the partial product has been multiplied by 6 to give the second line, because $42 = 6 \times 7$. Caution is needed here to set down the first figure of each answer in the right place. This can only be done by remembering the value of the multiplier in each case, and by applying the rule given in (75).

EXERCISE XXIV.

1. 27963×279; 85065×82; 406954×963.
2. 7401×567; 13472×39; 10968×62.
3. 54186×486; 7096×726; 2095×328.
4. 37264×714; 51386×497; 706258×369.

SECTION IV.—DESCENDING REDUCTION.

84. Reduction* is a process for finding how many quantities of one name are contained in a given concrete number of another name.

* It is evident that this Rule, which is usually placed far in advance in books of Arithmetic, has its proper logical position here. First, because no other rule than Multiplication is needed to perform the process; and Secondly, because no sum in Compound Long Division can ever be worked without Descending Reduction, or without assuming the principles here explained.

When it is required to reduce a quantity from a higher to a lower name the process is called *Descending* Reduction.

When it is required to reduce a quantity from a lower to a higher name the process is called *Ascending* Reduction.

85. Descending Reduction is performed by Multiplication.

Example I.—Let it be required to find how many farthings are contained in £57. This is simply a sum in Multiplication; for if we know how many farthings are in £1, 57 times that number will evidently give the number of farthings in £57. Or, if we do not know that 960 farthings make £1, but only that 4 farthings are 1 penny, 12 pence a shilling, and 20 shillings £1, the successive multiplication of £57 by 20, by 12, and by 4, will produce the same result.

£.
57
20
1140 = shillings in £57
21
13680 = pence in £57
4
54720 = farthings in £57

Because 20 shillings make £1, 20 times 57 or 1140 is the number of shillings in £57; and because 12 pence make 1s., 12 times 1140 or 13680 is the number of pence in 1140 shillings. So also, because 4 farthings make 1 penny, 4 times 13680 or 54720 is the number of farthings in 13680 pence. But 13680 pence = 1140 shillings or £57. Wherefore £57 = 54720 farthings.

Example II.—How many ounces are in 41 tons, and how many feet in 13 miles?

41 tons	13 miles
20	8
820 = cwts. in 4 tons	104 = furlongs in 13 miles
4	40
3280 = qrs. in 820 cwts.	4160 = poles in 104 furlongs
28	$5\frac{1}{2}$
26240	20800
6560	2080
91840 = lbs. in 3280 qrs.	22880 = yards in 4160 poles
16 lbs.	3
1469440 = oz. in 91840 or 41 tons	68640 = feet in 22880 yards

86. If the quantities proposed to be reduced are of different values, it is necessary to add each number to the line of figures representing the same kind as itself.

Example III.—Reduce £25 9s. $3\frac{1}{4}$d. to farthings.

$$
\begin{array}{r l}
\text{£. \ } s. \ d. & \\
25 \ \ 9 \ \ 3\tfrac{1}{4} & \\
\underline{20} \quad\quad\ & \\
509 & = \text{shillings in £25 9s. } (\overline{25 \times 20} + 9) \\
\underline{12} & \\
6111 & = \text{pence in 509 shillings and 3 pence } (\overline{509 \times 12} + 3) \\
\underline{4} & \\
24444 & = \text{farthings in 6111 pence} + 1 \text{ farthing } (\overline{6111 \times 4} + 1)
\end{array}
$$

RULE FOR DESCENDING REDUCTION.

87. Multiply by as many of the less as make one of the greater; add in to each line all the figures of the original number which refer to the same name, and continue the process, step by step, until the number is reduced to the required denomination.

Exercise XXV.

1. How many seconds in 4 lunar months? How many in 2 weeks 3 days? In 7 hours 45 minutes? From 5 P.M. on Tuesday to 7 A.M. on Saturday?

2. How many ounces in 7 cwt.? How many pounds in 3 tons 4 cwt.? How many drachms in 3 lbs. 13 oz.? How many grains in 5 oz. 11 dwt.?

3. How many feet in 4 miles? inches in 17 yds. 2 ft.? square feet in 4 acres? square yards in 3 square miles?

4. How many half-pints in 7 gallons? How many pecks in 19 quarters of corn? How many gills in a hogshead of spirits? How many pints in 12 hogsheads of ale?

SECTION V.—SIMPLE DIVISION.*

88. Division is a process—

I. For finding how many times one number contains another.

II. For separating a given quantity into a certain number of equal parts.

III. For finding a number which, if multiplied by another, will produce a third.

IV. For finding the multiplier, when the multiplicand and the product of a Multiplication sum are given.

89. The number to be divided is called the DIVIDEND; that which divides it the DIVISOR;† while the number which shows the times or parts of a time which the Dividend contains the Divisor is called the QUOTIENT.‡

Example.—Divide 30 by 5. Here the 30 is called the *Dividend*, because it is that which has to be divided. The 5 is called the *Divisor;* and the answer, when found, will be called the *Quotient.* This question may take several forms, *e.g.*:—

I. Find how many times 30 contains 5.

II. Separate 30 into 5 equal parts.

III. Find what number multiplied by 5 will give 30.

IV. If 30 be the product of two factors, and 5 be one of those factors, what is the other factor?

V. Divide 30 into as many parts as there are units in 5.

VI. Find a number which is as many times less than 30 as unity is less than 5.

VII. How many times can 5 be subtracted from 30? §

* *Preliminary Mental Exercise.*—Before beginning to work this Rule the student should be frequently exercised in giving the results of the Multiplication Table inversely. Thus: the 7th part of 35; the 9th of 108; the 4th of 32, &c., &c. This should be done until both processes become very familiar.

† From the Latin, *divido*, to divide.

‡ *Quoties* (Latin), how much, or, how many times.

§ It is important that a beginner should be accustomed to look at questions in

90. *Signs.* The sign used in Division is $\div$, thus: $12 \div 2 = 6$ is to be read 12 *divided by* 2 equals 6. The same is more frequently represented by placing the dividend above the divisor and drawing a line between them, *e.g.*: $\frac{20}{5} = 4$. In reading such a number, we commonly say 20 *by* 5 or 20 *upon* 5 equals 4.

91. Division is a short method of working Subtraction.

It was shown (60) that Multiplication was an abbreviated method of adding the same number to itself a certain number of times, but that Multiplication gives us no help in adding *different* numbers together. Similarly Division is of service when we wish to subtract the *same* number as often as we can from a greater number; but it will be of no use in any other subtractions. Thus: the sum just given might be worked in the following manner, and it would then be found that 5 can be subtracted from 30 six times in succession, and leave no remainder, and therefore that 30 contains 5 exactly 6 times:

	30	
First	5	
	25	First remainder.
Second	5	
	20	Second remainder.
Third	5	
	15	Third remainder.
Fourth	5	
	10	Fourth remainder.
Fifth	5	
	5	Fifth remainder.
Sixth	5	
	0	Sixth remainder.

Observation.—Whoever knows the Multiplication Table well, would, however, be able to tell at once that 6 is the multiplier which makes 5 into 30, and hence that 6 is the answer to the sum. A knowledge of the Multiplication Table is therefore as necessary in this as in the preceding Rule.

92. From (56) it appears that the multiplier in every Multiplication sum must always be an *abstract* number, because it expresses *how many times* the multiplicand is to be repeated, and cannot refer to a number of articles or things of any kind. It follows from this that in a Division sum either the divisor or the quotient must also be an abstract number, for one of those two numbers is always the multiplier which, if

---

ion from each of the points of view described in the text; and the teacher well to give many exercises on the Multiplication Table, varying the form of tion in the way described, before the pupil attempts to work out problems on a slate.

applied to the other, will give the dividend. It is not necessary that both divisor and quotient should be abstract, as only one of them corresponds to the multiplier of a Multiplication sum. The form of the question always determines which of the two is abstract. Thus: if we inquire how many times are £6 contained in £48, the quotient will be abstract; but if we ask what the 6th part of £48 is, the divisor (6) is the abstract number. In the former case, £6 is the multiplicand which, multiplied by 8, will give £48 as answer; and in the latter case £8 is the multiplicand which, multiplied by 6, the divisor, will produce the dividend. In both cases *one of the two must* be abstract.

93. Axiom IX.—*We divide one number by another, when we divide all the parts of the one by that other, and add the answers together.*

Demonstrative Example.—Divide 36 by 3: now $36 = 30 + 6$. If we first divide 30 by 3, and find that the answer is 10, and then divide 6 by 3, and obtain the result, 2, these two answers added together will give the quotient required, *i.e.*:—

$$\frac{36}{3} = \frac{30}{3} + \frac{6}{3} = 10 + 2 = 12.$$

Again, if 56 has to be divided by 4, the parts of 56 may be taken in any way we like, and thus it may be divided separately. Suppose we break up 56 into 20, 12, 16, and 8, which are its parts, then $\frac{56}{4} = \frac{20}{4} + \frac{12}{4} + \frac{16}{4} + \frac{8}{4} = 5 + 3 + 4 + 2 = 14$. This is not a convenient way; but still the answer might be found in this method, if there were any advantage in breaking up the number into those particular parts.

General Formula.—Whenever $a = b + c + d$

$$\text{Then } \frac{a}{x} = \frac{b}{x} + \frac{c}{x} + \frac{d}{x}$$

94. In Division we always proceed to separate the parts of the dividend in order, beginning with the highest. The right answer *might*, however, be found by beginning with the lower numbers in the dividend and dividing each of them into the required number of parts; but in most cases this would prove very inconvenient.

$$\begin{array}{r} 468(2 \\ \hline 234 \end{array}$$

In the sum, divide 468 by 2, it would be just as easy to begin at one end as another, thus: First take the half of 8, and set down 4; then the half of 60, and set down the 3 tens; and then the half of 400, which gives 2 for the hundreds place; but had the sum been, divide 568 by 2, we should, after taking the half of 8 and of 60, have to take the half of 500, which is 250, and then the sum would take this form:—

$$\begin{array}{rcl} 568(2 & & \\ \hline 4 & = & 8 \div 2 \\ 30 & = & 60 \div 2 \\ 250 & = & 500 \div 2 \\ \hline 284 & = & 568 \div 2. \end{array}$$

Here an Addition sum would have to be introduced, and the sum rendered longer than necessary. The ordinary method is as follows:—

Divide 293476 by 7.

$$\begin{array}{rcr} 7)293476 & & \\ \hline 40000 & = & 280000 \div 7 \\ 1000 & = & 7000 \div 7 \\ 900 & = & 6300 \div 7 \\ 20 & = & 140 \div 7 \\ 5 & = & 35 \div 7 \\ \frac{1}{7} & = & 1 \div 7 \\ \hline 41925\frac{1}{7} & = & 293476 \div 7 \end{array}$$

We here take the figures in order: first the 2 which, being a number of the 6th place, represents 200000, has to be divided by 7; but this will give no answer in the 6th place; 29 therefore, which is a number of the 5th place, is taken, and is found to contain seven 40000 times, and to leave 1 of that denomination, *i.e.*, 10 thousands, undivided; these 10 thousands, added to the 3 thousands, make 13000, the 7th part of which gives only 1 in the thousands place, and leaves 6000 still to be divided; and 6000 + 400 give 6400, which, divided by 7, gives 900, and leaves 100. This remainder, added to 70, makes 17 tens, the 7th of which is 2 tens, with a remainder, 3 tens; these, with 6 units, make 36, the 7th of which is 5, leaving 1 remainder. The 7th of this 1 cannot be found, so it is placed at the end of the sum thus $\frac{1}{7}$, and signifies that the division of this 1 has not been effected, but that the answer is 41925 *and the 7th part of* 1.

95. *Observation I.*—It will be seen that, as we ascertained at every step the greatest number which could come into that place of the answer, each figure as it was found might simply have been written without the ciphers in its proper place, and a great deal of trouble saved. Accordingly it is usual to omit all the numbers placed in the example except the last line. Beginners, however, should work out a few sums in this way.

Observation II.—Again, we did not take a 7th part of each of the numbers as they stood, but of the *nearest number which con-*

tained an exact number of sevens. Thus: the 7th of 280000, not of 290000, was first taken, because it gave an exact 4 in the 5th place, the remaining 10000 being carried forward and accounted for afterwards; so also the 7th of 7000, and not of 13000, was taken, the 6000 being carried on. Again, the 7th of 6400 could not be found in hundreds, so the 7th of 6300 was taken, and the remaining 100 carried. In like manner the 7th of 14 tens, and not of 17 tens, was next found; the 7th of 35 units, and not of 36 units. The whole dividend has therefore been resolved, for the purpose of dividing it by 7, into the following parts: 280000 + 7000 + 6300 + 140 + 35 + 1. Each of these parts contains an exact number of sevens, except the last, and all of them, if added together, would make up the whole dividend.

CASE I.—WHEN THE DIVISOR IS LESS THAN 13—

RULE.

96. Place the divisor on the left of the dividend (94), and find how many times it is contained in the first figure of the dividend, setting down the answer immediately under that figure. If it is not contained in the first, carry that number on, and join it to that on its right; then find how many times the divisor is contained in this number, carrying the remainder, if any, on to the next place, and setting down each figure of the quotient beneath that of the same value in the dividend. Whenever the divisor is not contained at all in the figures, write a cipher, to mark that that place is empty, and carry the remainder on as before.

Examples.—I. The complete process:—

Divide 729346 by 6.

$$
\begin{array}{rcrl}
6)\underline{729346} & & & \\
100000 & = & 600000 \div 6 & \text{leaving a remainder of } 100000 \\
20000 & = & 120000 \div 6 & \text{leaving no remainder} \\
1000 & = & 9000 \div 6 & \text{leaving a remainder of } \quad 3000 \\
500 & = & 3000 \div 6 & \text{leaving a remainder of } \quad\ \ 300 \\
50 & = & 300 \div 6 & \text{leaving a remainder of } \quad\quad 40 \\
7 & = & 42 \div 6 & \text{leaving a remainder of } \quad\quad\ \ 4 \\
\underline{\tfrac{4}{6}} & = & \underline{4 \div 6} & \\
121557\tfrac{4}{6} & = & 729346 \div 6 &
\end{array}
$$

II. The contracted or common process:—

$$6\underline{)729346}$$
$$121557\tfrac{4}{6} = 121557 \text{ and the 6th part of 4.}$$

Exercise XXVI.

(a). Work the following sums in Division in the manner described in (II.).

(b). Set down the first 8 sums as in (I.), showing what are the parts into which the dividend has been broken up, and adding them together to give the original dividend:—

1. Divide 72963 by 5; 847256 by 8; 124376 by 11; 12301 by 9; 547229 by 11; 729834 by 6; 271 by 5.

2. $\frac{72981}{7}$; $\frac{86945}{11}$; $\frac{302718}{9}$; $\frac{47216}{8}$; $\frac{82915}{5}$; $\frac{32917}{4}$.

3. $32916 \div 8$; $841729 \div 7$; $729186 \div 12$; $32741 \div 5$; $54729 \div 6$.

4. $\frac{274+8}{5}$; $\frac{56+37-9}{3}$; $\frac{4096+58+4}{7+2}$; $\frac{273+87}{16-5}$

97. Axiom X.—*We divide by the product of two or more numbers when we divide by each of them successively.*

Demonstrative Example.—If we first take the 7th part of a number, and then the 4th part of this quotient, we shall evidently have obtained the 28th part of the original number; for, as the 7th is contained 7 times in the whole, and the 4th of this 7th is contained 4 times in the 7th, this quantity is contained 4 times 7 times, or 28 times in the whole. Hence the 4th of a 7th is a 28th, and to divide by 4 and by 7 is to divide by 4 times 7, or 28. So, also, to divide by 3 and 5 is to divide by 15, &c.

General Formula.—If $a = b \times c$, then $\frac{x}{a} = \frac{x \div b}{c}$

98. Case II.—When a divisor is greater than 12, and its factors are known—

RULE.

Resolve the divisor into its factors, and divide by them in succession.

Example.—Divide 87423 by 56: now $56 = 7 \times 8$.

$$\begin{array}{l} 7)\underline{87423} \\ 8)\underline{12489} \quad = 87423 \div 7 \\ \quad 1561\tfrac{8}{56} = 87423 \div \overline{8 \times 7} \text{ or } 56. \end{array}$$

Exercise XXVII.

1. Divide 4362 by 28; 12475 by 45; 962842 by 84.
2. 27961 by 49; 378412 by 120; 23918 by 72.
3. 549 by 21; 62833 by 44; 12719 by 63; 8274 by 25.
4. 39628 by 18; 23718 by 32; 96247 by 99.
5. 82742 by 42; 39728 by 63; 82164 by 56.
6. 72910 by 84; 39721 by 48; 29714 by 108.

SECTION VI.—LONG DIVISION.

99. Many divisors cannot be resolved into factors, and many more could not be so resolved without much trouble. To divide by a number greater than 12, the following process is therefore employed:—

Divide 479632 by 28.

$$\begin{array}{rrr} 28) & 479632 & (10000 \\ & \underline{280000} & 7000 \\ 28) & 199632 & 100 \\ & \underline{196000} & 20 \\ 28) & 3632 & 9 \\ & \underline{2800} & \underline{\tfrac{20}{28}} \\ 28) & 832 & 17129\tfrac{20}{28} \\ & \underline{560} & \\ 28) & 272 & \\ & \underline{252} & \\ & 20 & \end{array}$$

Here we first take the figures, 47, which, standing in the fifth place, mean tens of thousands, and inquire how many times this number contains 28; the answer is *once*, but as the 47 are tens of thousands, we set down 10000 as part of the desired quotient; 19 tens of thousands are thus left undivided and are carried to the fourth place; with the 9 thousands which stand there they make 199 thousands, which divided by 28 gives us 7000, leaving a remainder, 3000, to be carried to the hundreds place. This 30 hundreds + 6 hundreds when divided by 28 gives 100 and leaves a remainder 8; this 800 or 80 tens added to 3 tens makes 83 tens, and this divided by 28 gives 2 tens for the quotient, leaving a remainder 27 tens; 27 tens or 270 if added to 2 give 272, which divided by 28 gives 9 and leaves a remainder 20. This last remainder cannot be divided by 28, so it is set down with

the mark of division under it ($\frac{20}{28}$), and this represents the 28th part of 20. The answers added together give 17129 and the 28th part of 20, or $17129\frac{20}{28}$.

100. *Observation.*—It is necessary again to notice here that the whole dividend has been broken up for convenience into a number of parts, and that each of these has been separately divided by 28. The parts have been chosen because each of them, except the last, contains an exact number of 28's; and the number of 28's thus contained in each part is separately set down as part of the quotient, and these partial quotients are added together at the end of the sum.

Now the parts into which the whole has been resolved are—

280000; 196000; 2800; 560; 252; and 20.

And because it has been ascertained that—

280000	contains 28	10000 times
196000	"	7000 "
2800	"	100 "
560	"	20 "
252	"	9 "
20	"	the 28th part of 20 times

Therefore 479632 contains 28 17129 and the 28th part of 20 times;

Or, the sum of these partial dividends contains 28 as many times as the sum of the several partial quotients.

RULE FOR LONG DIVISION.

101. Write the divisor to the left of the dividend and separate them by a line.

Take as many figures from the left of the dividend as there are in the divisor, or if this be a smaller number than the divisor take one more figure of the dividend; consider this separately as *the first partial dividend.* Find how many times it contains the divisor, place this number on the right as *the first partial quotient.* Multiply the divisor by it, and subtract this product from the first partial dividend, setting down only the remainder.

To this remainder add the next figure of the dividend. Find how many times this contains the divisor, and set down the number thus found as *the second partial quotient.* Multiply as before and subtract the new product from the second partial dividend, setting down the remainder only. To this remainder add the next figure of the dividend, and proceed as before.

102. *Observation.—The value or place of each figure in the partial quotient is the same as that of the right hand figure of the partial dividend to which it belongs.* Hence, if after bringing down one figure of the dividend, the partial dividend obtained is found not to contain the divisor, place a cipher in the quotient to show that a place is empty, bring down another figure of the dividend, and proceed as before.

Example. Divide 53946028 by 253. This may be set down in three ways: I. By showing the whole process, and stating the value of each line of figures as it is obtained. II. In the ordinary method, showing only the process of subtracting each product from the partial dividends. III. In the contracted method, in which only the several partial dividends are shown.*

I.

Analysis of Process.

		253) 53946028	(200000	=	50600000 ÷ 253
253 × 200000	=	50600000	10000	=	2530000 ÷ 253
		3346028	3000	=	759000 ÷ 253
253 × 10000	=	2530000	200	=	50600 ÷ 253
		816028	20	=	5060 ÷ 253
253 × 3000	=	759000	5	=	1265 ÷ 253
		57028	$\frac{103}{253}$	=	103 ÷ 253
253 × 200	=	50600	$213225\frac{103}{253}$	=	53946028 ÷ 253
		6428			
253 × 20	=	5060			
		1368			
253 × 5		1265			
		103			

Analysis of Quotient or Answer.

* When the principle of the Rule is understood, this method should invariably be employed in practice. It is evident that the several remainders written down here are the only figures required afterwards, and that writing the several products in order to subtract them is a waste of time and labour.

II. $253)53946028(213225\frac{103}{253}$

$$\begin{array}{r}
506 \\ \hline
334 \\
253 \\ \hline
816 \\
759 \\ \hline
570 \\
506 \\ \hline
642 \\
506 \\ \hline
1368 \\
1265 \\ \hline
103
\end{array}$$

III. $253)53946028(213225\frac{103}{253}$

$$\begin{array}{r}
334 \\
816 \\
570 \\
642 \\
1368 \\
103
\end{array}$$

Exercise XXVIII.

(a). Work the following sums as in Example III.:—

(b). Analyze the sums (in 1 and 2) in the manner described in Example I.

1. Divide 279684 by 507; 8920562 by 357; 72874 by 36; 509782 by 57; 723486 by 327; 5796843 by 126.

2. 4098765 by 78; 3250796 by 235; 8096274 by 427; 506897 by 525; 829704 by 18.

3. $72843 \div 65$; $786321 \div 49$; $52786 \div 4098$; $306845 \div 47$; $27839 \div 163$; $42745 \div 87$; $309628 \div 541$; $407864 \div 587$; $39721 \div 675$.

4. $5098 \div 18$; $672543 \div 29$; $458327 \div 123$; $50986 \div 372$; $278437 \div 526$; $209685 \div 324$.

5. Find a number equal to the expressions—

$$\frac{29864}{31}; \quad \frac{507968}{27}; \quad \frac{325+29+5}{47+4}; \quad \frac{62870-327}{5 \times 9}$$

6. $$\frac{2783 \times 6}{17}; \quad \frac{1293 \times (8+5)}{263}; \quad \frac{12086-119}{27 \times 3}$$

7. $$\frac{589+27+163-8}{25+19}; \quad \frac{31 \times 47}{14+7}$$

SECTION VII.—COMPOUND DIVISION, OR THE DIVISION OF CONCRETE QUANTITIES.

103. It has been seen that in Simple Division we begin with numbers of the highest value, divide them, and after getting the largest possible quotient in a number of the same value, carry on the remainder, reduce it from thousands into hundreds, or from hundreds to tens, and then find the greatest quotient in this lower name. So if instead of thousands, hundreds, tens, and units, the dividend consisted of pounds, shillings, pence, and farthings, it would be necessary to find the quotient in pounds, then carry on the remainder and reduce them to shillings, afterwards to divide the shillings and carry on the remainder to the pence, and so on. (See Rule for Descending Reduction.) In the following example it will be seen that the two processes are really the same in principle :—

Divide 23733 by 6, and also £23 7s. $3\frac{3}{4}$d. by 6.

6) 23733 (3 thousands
18
5 thousands
10
6) 57 (9 hundreds
54
3 hundreds
10
6) 33 (5 tens
30
3 tens
10
6) 33 (5 units
30
3 remainder

Answer—3 thousands, 9 hundreds, 5 tens, 5 units, and the sixth part of 3; or $3955\frac{1}{2}$.

£. *s.* *d.*
6) 23 7 $3\frac{3}{4}$ (3 pounds
18
5 pounds
20
6) 107 (17 shillings
102
5 shillings
12
6) 63 (10 pence
60
3 pence
4
6) 15 (2 farthings
12
3 farthings remainder

Answer—3 pounds, 17 shillings, 10 pence, 2 farthings, and the sixth part of 3 farthings; or £3 17s. 10d. $2\frac{1}{2}$ farthings.

104. Although we do not in practice set down 10 as a multiplier, as shown in the former of these examples, yet we mentally make the same reduction as if we did so. Thus, after finding the 6th of 23 thousands to give a quotient 3 thousands and leave 5 thousands remainder, we bring down the other figure to its side and treat the new number 57 by itself. But this 57 is of a lower name than thousands; the 5 thousands have really been converted into 50 hundreds and added to the other hundreds of the dividend. The next remainder, 3 hundreds, is reduced in like manner to 30 tens and added to the 3 tens which already form part of the dividend.

This method of reducing the remainder of each denomination or place into equivalent numbers in the lower name or place, is employed throughout the whole of Simple Division, and is equally applicable to Compound. In the second of the examples the number of pounds remaining after the first quotient is obtained is reduced to shillings and added to the other shillings of the dividend. A new division is then made of shillings only, the remainder reduced to pence and treated in the same way.

RULE FOR COMPOUND DIVISION.

105. Divide the numbers of the highest name as in Simple Division. By the Rule of Reduction (87) bring the remainder and the next figure into equivalent numbers of the next lower name. Make a new Division sum with this number, and set down the answer by itself. Reduce the remainder into numbers one step lower, and divide this new number separately. Proceed in the same way with the remainders until the lowest is reached.

106. *Observation.*—No new truth of Arithmetic is wanted or illustrated here. Axiom IX. tells us that it is allowable to break up the dividend in any manner which may be most convenient, and to divide it part by part. And it has been seen (94) that it is more convenient to begin with the greater numbers.

Example.—What is the fourteenth part of 23 tons 17 cwt. 3 qrs. 5 lbs.?

I. The entire process.

tons cwt. qrs. lbs.

14)23 17 3 5(1 ton = 14 tons ÷ 14
14
9 tons
20
14)197(14 cwt. = 196 cwt. ÷ 14
196
1 cwt.
4
14) 7 qrs.(0 qrs.
28
14)201(14 lbs. = 196 lbs. ÷ 14
196
5 lbs.
16
14) 90(6 oz. = 84 oz. ÷ 14
84
6 — $\frac{6}{14}$ oz. = 6 oz. ÷ 14

II. The ordinary or contracted process.

tons cwt. qrs. lbs.

14)23 17 3 5 (1 „ 14 „ 0 „ 14 „ $6\frac{6}{14}$ oz.
9
20
197
1
4
7
28
201
5
16
90
6

Answer.

1 ton 14 cwt. 14 lbs. $6\frac{6}{14}$ oz. = 23 ton 17 cwt. 3 qrs. 5 lbs. ÷ 14.

Exercise XXIX.

(a). Work the following sums :—

(b). Analyze the sums in (2), (7), (10), as in Example I.

1. Divide £723 11s. 8d. by 4 ; £52 10s. $4\frac{1}{2}$d. by 6 ; £293 10s. $8\frac{1}{4}$d. by 3.

2. Find the 5th of £297 16s. 3d. ; the 6th of £274 13s. 8d. ; and the 7th of £1000.

3. Find the 7th, the 8th, and the 9th of £6274 15s. $8\frac{1}{2}$d.

4. £297 14s. $3\frac{1}{4}$d. ÷ 28 ; £3654 10s. 11d. ÷ 352.

5. £1270 3s. 7d. ÷ 514 ; £1560 3s. ÷ 17 ; £2740 3s. $1\frac{1}{4}$d. ÷ 510.

6. £12783 13s. 9d. ÷ 60 ; £3185 3s. $6\frac{1}{2}$d. ÷ 137 ; £2186 15s. 3d. ÷ 49.

7. £29841 13s. 7½d. ÷ 631; £62108 17s. 9¼d. ÷ 65; £270 10s. ÷ 135.

8. 73 cwt. 1 qr. 26 lbs. ÷ 15; 21 tons 5 cwt. 3 qrs. ÷ 29; 7 cwt. 25 lbs. ÷ 5.

9. 6 lbs. 3 oz. 4 dwt. ÷ 8; 15 lbs. 11 oz. 7 grs. ÷ 14; 29 lbs. 3 dwt. 14 grs. ÷ 125.

10. 17 miles 2 fur. 17 poles ÷ 16; 192 miles 6 fur. ÷ 23; 7 fur. 18 poles 3 yds. ÷ 14.

11. Find a 4th, a 6th, an 11th, and a 13th of a field of 17 acres.

12. 372 qrs. 2 bush. 1 peck ÷ 8; 17 qrs. 7 bush. 3 pecks ÷ 27; 15 qrs. 6 gals. ÷ 9.

107. In Compound Division the divisor is generally abstract, and the quotient consists of concrete numbers of the same kind as the dividend. It is evident (89) that the word quotient does not in strictness apply to such an answer as this.

108. Sometimes, however, a concrete number has to be divided by another similar concrete number.

Example.—How many times is 2¼d. contained in £43 18s. 7d.? Here the answer will be an abstract number.

Now because, by Rule of Descending Reduction,
2¼d. contains 9 farthings, and £43 8s. 7d. contains 42172 farthings,
The question takes this form:—

How many times are 9 farthings contained in 42172 farthings?

Or, how many times are 9 contained in 42172?

$$\text{By Simple Division } \frac{42172}{9} = 4685\tfrac{7}{9} = \text{the answer.}$$

109. *Observation.*—It is evident that the two numbers must refer to magnitudes of the same name. It would be absurd to ask how many times 3 miles were contained in 17 days.

It should also be noticed that although we may divide one concrete number by another, we cannot multiply one concrete number by another. The question, for example, Multiply £17 3s. by £17 3s. is evidently unmeaning. For of two factors the multiplier must *always* be an abstract number (56), but of divisor and quotient, either may be concrete, provided the other is abstract.

RULE FOR DIVIDING ONE CONCRETE NUMBER BY ANOTHER OF THE SAME KIND.

110. Reduce both to the same name (the highest to which both can be reduced) and divide the greater by the less.

EXERCISE XXX.

1. £17 3s. 9d. ÷ 3¾d.; £965 14s. 3½d. ÷ 2s. 6¼d.; £87 4s. 1½d ÷ 7½d.

2. £392 10s. 3½d. ÷ 1s. 7¼d.; £62 10s. ÷ 1s. 3d.; £274 18s. 5d. ÷ 4¾d.

3. £1027 5s. 6d. ÷ £2 3s. 8d.; £453 12s. 8¼d. + 2s. 9¼d.; £62 3s. 5d. ÷ 1s. 8½d.

4. 23 lbs. 15 oz. ÷ 2 oz. 8 drs.; 17 cwt. 3 qrs. ÷ 17 lbs. 3 oz.; 196 cwt. ÷ 2 lbs. 11 oz.

5. 17 miles 3 fur. 18 poles ÷ 29 yds.; 23 miles 6 fur. ÷ 6 yds. 2 ft.; 412 miles 6 fur. 18 poles ÷ 3 miles 2 fur.

6. 5 years 186 days ÷ 17 hours 3 mins.; 27 weeks 4 days ÷ 3 days 6 hours.

7. 17 acres 3 roods 29 poles ÷ 2 roods 7 poles; 5166 acres 2 roods 14 poles ÷ 5 acres 3 roods; 29 acres 1 rood 13 poles ÷ 39 poles.

8. 17 bush. 2 pecks ÷ 5 pints; 728 qrs. 3 bush. 1 peck ÷ 2 qrs. 3 pecks; 2974 qrs. ÷ 1 bush. 1 peck.

9. How many coins worth 2¾d. each are worth £25?

10. How many fourpenny pieces and sixpences are there in £28 17s. 6d.?

11. How many guineas are in £7934?

12. Reduce £1738 to half-guineas, and the same number of half-guineas to pounds.

13. How many packages, each weighing 11 oz., can be made from 2 qrs. 18 lbs.?

14. How many allotments of 18 poles each can be made from 19 acres 3 roods?

15. How often will a hoop 2 ft. 3 in. in circumference revolve in rolling 3 miles?

METHODS OF PROVING MULTIPLICATION AND DIVISION.

111. The connexion already explained (88) between Multiplication and Division shows that either rule may be used to prove the other. For because the answer to a Multiplication sum is produced by multiplying a given number by a second, it follows that to divide this answer by the second is to reproduce the original multiplicand.

TO PROVE MULTIPLICATION—RULE.

112. Divide the answer by the multiplier, and if it gives the multiplicand the sum is right.

EXERCISE XXXI.

Prove the first twelve sums in Exercise XVI. by this rule.

A similar method applies to Division, because (88) the quotient sought is always such a number that if it be used as a multiplier and applied to the divisor it will produce the dividend. The remainder however, if any, being *that portion of the dividend which continues undivided*, must be added to the product of divisor and quotient.

TO PROVE DIVISION—RULE I.

113. Multiply the quotient by the divisor, add the remainder to the result, and if the answer equals the dividend the sum is right.

EXERCISE XXXII.

Prove the first 12 sums in Exercise XXVI. (3 and 5) by this method.

TO PROVE DIVISION—RULE II.

114. Select from the sum, after working it, all the partial dividends which have been taken, including the remainder. Add these together, and if the answer equals the dividend the sum is right.

Observation.—This has been already explained (100) as a process of analysis, it is also a method of proof.

EXERCISE XXXIII.

Prove sums in Exercise XXVI. (2 and 4) by this method.

115. *Casting out nines.* The reason of this Rule will be found in the chapter on properties of numbers in the Appendix.

I. To prove Multiplication.

```
 7963    7
   58    4
 ------  --
 63704   28 = 9 × 3 + 1
39815
 ------
461854    1
```

Add up the figures in the multiplicand, divide the sum by 9 and set down the remainder. (In Example, $7+9+6+3=25=2\times 9+7$, *set down* 7). Do the same with the multiplier. (In Example, $5+8=13=9+4$, *set down* 4). Do the same also with the answer to the sum ($4+6+1+8+5+4=28=3\times 9+1$, *set down* 1). Now, if the product of the two former remainders (7 and 4) divided by 9 leaves this last remainder (1), the sum is likely to be right; but if not it is certainly wrong.

EXERCISE XXXIV.

Prove the first 12 sums in Exercise XXVII. by this method.

II. To prove Division.

```
 2             6
29) 683475 (23568
     103
      164
       197
        235
          3   30
```

Add up the figures in the divisor and quotient, divide each by 9 and set down the remainders over them. (In Example, $2+9=11$, set down 2 over divisor, and $2+3+5+6+8=24=2\times 9+6$, set down 6 over quotient.) Then subtract the remainder from the dividend and add up the figures which are left. (In Example, $6+8+3+4+7+5-3=30$.) Divide this by 9, and if the remainder is the same as that obtained by dividing the product of the other two numbers by 9 the answer is likely to be right; but if not it is certainly wrong. (In Example, $2\times 6=9+\underline{3}$, and $30=3\times 9+\underline{3}$.)

EXERCISE XXXV.

Prove sums in Exercise XXVIII. (1 and 2) by this method.

SECTION VIII.—APPLICATION OF DIVISION TO REDUCTION.

116. The reduction of any concrete number to an equivalent number of a higher name is called Ascending Reduction.

Ascending Reduction is to be worked by Division.

Example.—How many days are in 1000000 seconds? Here, if we know the number of seconds in a day, and divide 1000000 by it, we may at once ascertain the number of days. The same result may be reached by successive steps, thus: Because there are 60 seconds in one minute, therefore there are one-sixtieth part of 1000000 minutes in 1000000 seconds. Hence 16666 minutes and 40 seconds = 1000000 seconds. But because there are 60 minutes in one hour, there are one-sixtieth part of 16666 hours in 16666 minutes. Hence 277 hours 46 minutes = 16666 minutes. But again, because there are 24 hours in a day, there are one-twenty-fourth of 277 days in 277 hours. Hence 277 hours = 11 days 13 hours.

```
60)1000000
 60)16666  40
  24)277   46
      11   13
```

Therefore we have found successively that

1000000 seconds = 16666 minutes 40 seconds.

1000000 seconds = 277 hours 46 minutes 40 seconds.

1000000 seconds = 11 days 13 hours 46 minutes 40 seconds.

RULE FOR ASCENDING REDUCTION.

Divide the number by as many of the less quantity as make one of the next greater. Leave the remainder, and divide the first quotient by as many of its denomination as make one of the next above it. Leave the remainder and continue to divide each new quotient until the highest is reached.

Observation.—The value of each remainder *is that of the dividend from which it is left*, and each quotient has the value of the denomination just above that of the dividend from which it is obtained.

Example.—Reduce 2534 farthings into pounds, and 4398 grains into pounds troy.

$$\begin{array}{l} 4)\underline{2534} \text{ farthings} \\ 12)\underline{633} \quad 2 = 633 \text{ pence} + 2 \text{ farthings} \\ 20)\underline{52} \quad 9 = 52 \text{ shillings } 9 \text{ pence} \\ \quad 2 \; 12 = 2 \text{ pounds } 12 \text{ shillings} \\ \therefore 2534 \text{ farthings} = £2\ 12s.\ 9\tfrac{1}{2}d. \end{array}$$

$$\begin{array}{l} 24)\underline{4398} \text{ grains} \\ 20)\underline{183} \quad 6 = 183 \text{ dwt. } 6 \text{ grs.} \\ \quad 9 \quad 3 = 9 \text{ oz.} + 3 \text{ dwt.} \\ \therefore 4398 \text{ grains} = 9 \text{ oz. } 3 \text{ dwt. } 6 \text{ grs.} \end{array}$$

Exercise XXXVI.

1. Reduce 1233 farthings to pounds, and 171923 halfpence to pounds.
2. How many pounds in 27963 farthings, and in the same number of halfpence and pence?
3. Reduce 23856 seconds to hours; 158960 minutes to days; and 10005000 hours to years.
4. How many crowns in 583721 farthings?
5. Reduce 270628 feet to miles, and 863 inches to yards.
6. What is the number of tons in 447826 ounces?
7. By how much does the number of gallons in 10000 pints exceed the number of hogsheads in 10000 gallons?
8. Reduce 7098 grains to Troy pounds, and the same number to Avoirdupoise pounds.

** SECTION IX.—MULTIPLICATION AND DIVISION.

118. Multiplication.—The relations between multiplier, multiplicand, and product, in the former of these rules, and between the corresponding terms, quotient, divisor, and dividend, in the latter rule, are so important that it is necessary, before proceeding further in arithmetic, to examine them somewhat more closely.*

119. *Definitions.*—When a number is multiplied by itself the product is called the *second power* of that number. If this product be again multiplied by the first, the new product is the *third power*. Thus, the product of eight twos is the eighth power of 2, and so on.

This is expressed thus, $2^8 = 2$ to the 8th power.

So also, $a^5 = a \times a \times a \times a \times a = a$ raised to the fifth power, or the product of *five* a's.

* For the right understanding of the more advanced portions of the science, the axioms here stated are indispensable, and they will frequently be alluded to in future demonstrations.

The number which indicates the power to which a number is raised is called the Exponent. Thus in the expressions, 10^2, a^6, x^m, 2, 6, and m are the exponents.

Observation.—All the numbers represented by 1 followed by ciphers, are powers of 10. Thus $10 = 10^1$, $100 = 10^2$, $1000 = 10^3$, $10000 = 10^4$, $100000 = 10^5$, &c. By counting the ciphers we get the exponent of the 10, *i.e.*, we learn what power of 10 the number represents.

120. Axiom XI.—*We increase or diminish the product when we increase or diminish the multiplier.*

Demonstrative Example.—If we have a number, say 26, to be multiplied by 6, and obtain a certain answer, it is evident that multiplying by three times 6 would give three times that answer.

Again, if instead of multiplying by 6 we multiplied by the 3rd part of 6, we should obtain only one-third of the product.

General Formula.—If $a \times b = c$, then $a \times mb = mc$

$$\text{and } a \times \frac{b}{m} = \frac{c}{m}.$$

121. Axiom XII.—*We increase or diminish the product when we increase or diminish the multiplicand.*

Demonstrative Example.—Suppose on multiplying one number by a second we obtain a certain answer, multiplying ten times that number by the second would give ten times that answer; and similarly, multiplying a tenth part of that number by the second would give only a tenth part of that answer.

General Formula.—If $a \times b = x$, then $na \times b = nx$

$$\text{and } \frac{a}{n} \times b = \frac{x}{n}.$$

The result of the two last articles may be generally expressed thus:—

122. *If we increase or diminish either of the factors a certain number of times we make the same increase or diminution in the product.* This is sometimes expressed by saying that the product *varies as* the multiplicand or multiplier.

123. *If one factor is increased as many times as another is diminished the product remains unaltered.*

Demonstrative Example.—If instead of multiplying 8 by 6 we multiply twice 8 by the half of 6, or if we multiply three times 8 by the third of 6, we obtain exactly the same answer. For the increase of the one factor, which tends to make the product a certain number

of times greater, destroys the effect of the decrease of the other, which tends to make the product the same number of times less.*

Thus 7 times 12 $= (3 \times 7) \times \frac{12}{3}$ or $(4 \times 7) \times \frac{12}{4}$.†

General Formula.—$(a \times b) = na \times \frac{b}{n}$.

EXERCISE XXXVII.

Place a factor in each of the parentheses so that the products shall be equal.

$5 \times 9 = \frac{9}{3} \times (\ \ )$; $16 \times 12 = (\ \ ) \times \frac{12}{4}$; $49 \times 3 = \frac{49}{7} \times (\ \ )$.

$18 \times 6 = \overline{2 \times 6} \times (\ \ )$; $36 \times 8 = \frac{36}{4} \times (\ \ )$; $350 \times 17 = \frac{350}{10} \times (\ \ )$.

$a \times b = 3a \times (\ \ )$; $w \times y = \frac{w}{a}(\ \ )$; $p \times q = np \times (\ \ )$.

124. *Corollary I.—If there be three numbers so related that the first is as many times greater or less than the second, as that second is greater or less than the third, the product of the first and third will equal the second multiplied by itself.*

This is only a particular application of the truth expressed in (123).

Thus: 7, 14, 28. Here $14 = 2 \times 7$, and $28 = 2 \times 14$. But (123) 14 times $14 =$ the half of $14 \times$ twice 14, or 7×28.

Again, in the numbers 3, 12, 48. Because 12 is as many times more than 3 as it is less than 48; therefore $12 \times 12 = 3 \times 48$, and generally

If a is as many times greater than b, as b is greater than c } Then ac
or,
If a is as many times less than b, as b is less than c...... } $= bb$.

* The principle here explained suggests to us another method of proving Multiplication, although it is too cumbersome to be often used. Suppose I have 586 to multiply by 36, and after doing it wish to verify the answer; twice 586, or 1172, multiplied by the half of 36, or 18, would produce the same result; so would 6 times 586 multiplied by the *sixth* part of 36.

† Although the form employed here ($\frac{12}{4}$) appears fractional, it is necessary to remind teachers that we are here only concerned with it as far as it implies *division*. It will of course be read "the fourth part of 12." Throughout the whole of this chapter similar expressions should be read in the same way. The teacher will see that no one definition or principle of "Fractions" is anticipated by this arrangement, although when a pupil arrives at Fractions it will be a great advantage to him to be already familiar with such forms under the name of divisor and dividend.

EXERCISE XXXVIII.

Place a number in each parenthesis so that the products of the first and third shall equal the second multiplied by itself.

1. 3, 12 ( ); 2, 14 ( ); 18, 6 ( ).
2. 36, 18 ( ); 27, 9 ( ); 7, 21 ( ).
3. 25, 5 ( ); 72, 144 ( ); 18, 54 ( ).
4. a, $3a$ ( ); x, nx ( ); np, p ( ).

125. Whenever the product of two numbers is known to equal the product of two others, and any 3 of these four numbers being given we are required to find the fourth, we must find the product of the one pair which is given and divide by the remaining one.

Because $16 \times 3 = 4 \times 12$, the 16th part of 4×12 will equal 3; the 3rd part of 4×12 will give 16; the 4th part of 16×3 will give 12; and the 12th part of 16×3 will give 4. If either of these had been unknown it might have been found thus:—

$$12 = \frac{16 \times 3}{4}; \quad 4 = \frac{16 \times 3}{12}; \quad 16 = \frac{4 \times 12}{3}; \quad 3 = \frac{4 \times 12}{16}.$$

General Formula.—Whenever $ab = cd$

$$\text{Then } a = \frac{cd}{b}; \quad b = \frac{cd}{a}; \quad c = \frac{ab}{d}; \quad d = \frac{ab}{c}.$$

EXERCISE XXXIX.

Place a number in each parenthesis so that the products shall be equal in each of the following cases.

1. $16 \times 8 = 2 \times ( )$; $27 \times ( ) = 4 \times 9$.
2. $81 \times 4 = 12 \times ( )$; $542 \times 36 = 391 \times ( )$.
3. $4726 \times 9 = 27 \times ( )$; $5089 \times 63 = 716 \times ( )$.
4. $16 \times 7192 = 314 \times ( )$; $154 \times 379 = 812 \times ( )$.
5. $7246 \times 372 = 128 ( )$; $100 \times 5 = 75 \times ( )$.
6. $36 \times 81 = 243 \times ( )$; $a \times b = c \times ( )$.

126. *Corollary II.—If there be four numbers so related that the first is as many times more or less than the second, as the third is more or less than the fourth, the product of the second and third will equal that of the first and fourth.*

For in this case the second and third would give a certain product; but of the other two one is as many times greater than the second as the other is less than the third, or one is as many times less than the second as the other is greater than the third, hence (123) the two products must be equal.

Example.—5, 20, 3, 12. Here the second equals 4 times the first, and the fourth 4 times the third.

So because of the numbers 5, 20, 3, 12, the second is as many times greater than the first as the third is less than the fourth; $20 \times 3 = 5 \times 12$, *i.e.*, 20 times 3 = the fourth of 20×4 times 3.

Exercise XL.

Place a number in the vacant places in each of the following exercises, such that if multiplied by that which is linked to it it will equal the product of the other two.

1.	5	15	7	( )	2.	5	20	16	( )
3.	10	( )	4	12	4.	90	( )	18	2
5.	17	( )	9	18	6.	$3a$	a	b	( )
7.	25	5	75	( )	8.	$\frac{x}{4}$	x	y	( )
9.	pq	p	( )	m	10.	( )	b	a	$n \times a$

It may easily be inferred from the last paragraph that—

127. *If there be a series of numbers, arranged in order by regular and equal multiplication or division, they may be formed into a number of pairs of factors, each pair giving the same product.*

Thus, in the following series, in which each number is 3 times that on its left,

12 36 108 324 972 2916

108 multiplied by 324 would give a certain product. But $36 = \frac{108}{3}$ and $972 = 108 \times 3$. $\therefore$ 108×324 = the third of 108×3 times $324 = 36 \times 972$. So also 12 is as many times less than 108 as 2916 is more than 324; therefore $12 \times 2916 = 108 \times 324$.

In the same manner any two numbers may be taken in such a series, and their product will be found equal to that of any other two at equal distances from the first two.

128. From (125) we infer that the product of any two numbers at equal distances from any *one* of such a series will equal the second power of that one. For by the hypothesis every figure on one side of it will be as many times less than the number as that in the corresponding place on the other side will be greater; hence the

product of the two numbers so chosen will be equal to the square of the middle one.

Exercise XLI.

Find how many equal products can be made out of either of the following series:—

1. 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024.
2. 3, 6, 12, 24, 48, 96, 192.
3. 1, 5, 25, 125, 625, 3125, 15625.
4. 32, 16, 8, 4, 2, 1, $\frac{1}{2}$.

129. Division.—It is necessary to remember that in every Division sum the dividend is the product of which the divisor and the quotient are the two factors, and that because

In multiplication, multiplier $\times$ multiplicand $=$ product;
In division, quotient $\times$ divisor $=$ dividend;

Therefore every truth which could be asserted of the parts of a Multiplication sum can be asserted in another form which will be applicable to Division.

130. Axiom XIII.—*If we increase or diminish the dividend a certain number of times, we make the same alteration in the quotient.*

Demonstrative Example.—If 28 divided by 4 will give a certain answer, 3 times 28 divided by 4 will give 3 times that answer, and the half of 28 divided by 4 will give the half of that answer; or,

General Formula.—If $\frac{a}{b} = x$, $\frac{na}{b} = nx$, and $\frac{a \div n}{b} = \frac{x}{n}$.

Observation.—This is the converse of that in (122). It was there stated that increasing one of the factors increased the product; here it is asserted that increasing the product makes necessary the increase of one of the factors. For the quotient being one of the factors, and the other (the divisor) remaining the same, whatever increases the product (the dividend) makes a corresponding increase necessary in the other factor. Similarly it may be seen that if the dividend has been diminished while the divisor remains the same, the other factor (the quotient) must also be diminished as much as the dividend has been. This is sometimes expressed by saying that the quotient *varies as* the dividend.

131. Axiom XIV.—*If the divisor be increased a certain number of times, the quotient is diminished in the same degree; but if the divisor be diminished, the quotient is increased.*

Demonstrative Example.—If 270 divided by 15 gives a certain answer, 270 divided by twice 15 would give one-half of that answer; and 270 divided by a fifth of 15 would give 5 times that answer.

General Formula.—If $\frac{a}{b} = x$, then $\frac{a}{nb} = \frac{x}{n}$, and $\frac{a}{b \div n} = nx$.

Observation.—The assertion in (123) that if one factor be increased while the other is diminished the product remains unaltered, was seen to be true in all cases. Its converse is equally true, that if, while the product remains the same, one of the factors be increased the other must be diminished, and if one be diminished the other must be increased. But in Division sums the dividend is the product and the quotient and divisor are the factors. Hence with the same dividend and a smaller divisor we have a greater quotient, and with the same dividend and a greater divisor a smaller quotient.

The truth contained in the two last axioms may be concisely expressed thus:—

132. The answer to a Division sum is made greater by increasing the dividend or by diminishing the divisor; but it is made less by diminishing the dividend or by increasing the divisor.*

133. Axiom XV.—*If the dividend and divisor be either both increased or both diminished the same number of times, the quotient remains unaltered.*

Demonstrative Example.—If 42 divided by 6 gives 7, twice 42 divided by twice 6 will give the same answer; ten times 42 divided by ten times 6 will give the same answer; as also the half of 42 divided by the half of 6; or the third of 42 divided by the third of 6, &c., &c. So—

General Formula.—If $\frac{a}{b} = x$, $\frac{an}{bn} = x$, and $\frac{a \div n}{b \div n} = x$.

Observation.—This axiom depends on the same considerations as (122). It was there stated if one factor remained the same while the other was increased or diminished, the product was increased or diminished in the same degree. It is equally evident that if the product and one factor are both increased or both diminished in the same degree, the other factor may remain the same. But in every Division sum the dividend is the product of which the divisor is one factor and the quotient the other.

* In the higher arithmetic it would be thus enunciated:—The quotient varies directly as the dividend and inversely as the divisor.

Exercise XLII.

Find three other pairs of divisors and dividends which would give the same quotient as each of the following:—

1. $\frac{290}{10}$; $47 \div 3$; $29 \div 6$.
2. $58 \div 4$; $572 \div 12$; $435 \div 8$.
3. $48 \div 6$; $59 \div 12$; $523 \div 6$.
4. $15 \div 3$; $498 \div 20$; $135 \div 16$.

This principle is practically applied in the following cases:—

134. Case I.—When both dividend and divisor terminate in one or more ciphers—

RULE.

Cut off the same number of ciphers from both, and work the division with the numbers thus diminished.

For to cut off a cipher is to divide by 10; to cut off two, three, or four ciphers is to divide by the second, third, or fourth power of ten. But if this be done to both, the sum will be more readily worked, and the quotient will be the same as if the numbers had not been altered.

Exercise XLIII.

Work the following sums by the contracted method:—

1. $58600 \div 700$; $273000 \div 18000$; $396180 \div 500$.
2. $3500 \div 400$; $5169000 \div 800$; $37290 \div 90$.
3. $7200 \div 100$; $51980 \div 60$; $573000 \div 27000$.

135. Case II.—When the divisor is not a whole number—

RULE.

Multiply both divisor and dividend by such a number as will convert the divisor into a whole number, and then perform the division.

Example.—Divide 573 by $3\frac{3}{4}$.

Here we choose 4 because it is the multiplier which, applied to $\frac{3}{4}$, will convert it into a whole number.

$$(573 \times 4) \div (3\tfrac{3}{4} \times 4) = 573 \div 3\tfrac{3}{4},$$

But $573 \times 4 = 2292$, and $3\frac{3}{4} \times 4 = 15$.

$$\therefore \frac{573}{3\frac{3}{4}} = \frac{2292}{15} = 152\tfrac{12}{15}.$$

EXERCISE XLIV.

Work the following sums :—

1. $463 \div 2\frac{1}{3}$; $827 \div 5\frac{1}{3}$; $1006 \div 8\frac{1}{2}$.
2. $587 \div 4\frac{1}{3}$; $327 \div 2\frac{2}{3}$; $789 \div 5\frac{3}{4}$.
3. $6908 \div 2\frac{1}{4}$; $5172 \div 4\frac{1}{2}$; $3976 \div 6\frac{1}{4}$.
4. $12506 \div 3\frac{1}{2}$; $1089 \div 5\frac{1}{2}$; $1627 \div 7\frac{2}{3}$.
5. $1396 \div 4\frac{5}{8}$; $4096 \div 3\frac{3}{8}$; $8162 \div 4\frac{1}{8}$.

136. CASE III.—WHENEVER THE PRODUCT OF ANY SET OF NUMBERS HAS TO BE DIVIDED BY THE PRODUCT OF ANOTHER SET, AND THE TWO SETS CONTAIN COMMON FACTORS—

RULE.

Strike out the factors common to dividend and divisor, and perform the division with the remaining products only.

This operation is called *Cancelling*.

Example.—Divide $75 \times 6 \times 8 \times 4$ by $5 \times 9 \times 8 \times 6$. Here the 8 and 6 occur in both, and 75 may be resolved into 15×5. Hence 8 and 6 and 5 may be rejected, because we thus divide both dividend and divisor by the same numbers, and (133) this does not alter their quotient. Hence,

$$\frac{75 \times 6 \times 8 \times 4}{5 \times 9 \times 8 \times 6} = \frac{75 \times 4}{5 \times 9} = \frac{(15 \times 5) \times 4}{5 \times 9} = \frac{15 \times 4}{9}.$$

EXERCISE XLV.

Simplify the following by cancelling, and find the answers :—

1. $\dfrac{5 \times 4 \times 3}{9 \times 6 \times 5}$; $\dfrac{28 \times 12}{7 \times 12}$; $\dfrac{27 \times 4 \times 9}{3 \times 8 \times 9}$.

2. $(214 \times 6) \div (27 \times 6)$; $(23 \times 5 \times 108) \div (4 \times 5 \times 12)$.

3. $\dfrac{7 \times 8 \times 9}{3 \times 8 \times 6}$; $\dfrac{20 \times 12 \times 48}{24 \times 8 \times 10}$; $\dfrac{b \times c \times d}{a \times x \times b \times c}$.

137. Suppose it is required to multiply 6^4 by 6^3. Now (119) 6^4 is $6 \times 6 \times 6 \times 6$, or the product of 4 sixes, and $6^3 = 6 \times 6 \times 6$ or the product of 3 sixes. Hence $6^4 \times 6^3 = (6 \times 6 \times 6 \times 6) \times (6 \times 6 \times 6) = 6^7$, or the product of 7 sixes. The same principle applies to all similar cases. Hence we find the product of different powers of the same number by adding the exponents of those powers. This may be generally expressed—

Example.—$x^5 \times x^4 = x^{5+4} = x^9$, and $a^m \times a^n = a^{m+n}$.

The operation of division depends on the same considerations. Thus to divide 3^5 by 3^2 is to divide the product of 5 threes by the product of 2 threes.

$$\frac{3^5}{3^2} = \frac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3} = 3^{5-2} = 3^3.$$

Hence whenever a certain power of a number has to be divided by the same number raised to some other power, we must subtract the exponent of the divisor from that of the dividend.

Example.—$x^5 \div x^2 = x^{5-2} = x^3$.

$$a^m \div a^n = a^{m-n}.$$

Exercise XLVI.

Simplify the following expressions:—

1. $8^3 \times 8^5$; $12^7 \times 12^6 \times 12^4$; $15^3 \times 15^2 \times 15$.
2. $a^6 \times a^7$; $p^3 \times p^2 \times p$; $q^m \times q^n \times q^p$.
3. $\dfrac{6^8 \times 6^4}{6^3}$; $\dfrac{15^4 \times 15^9 \times 15^2}{15^7 \times 15}$; $\dfrac{3^4 \times 7}{3^2 \times 5}$.
4. $\dfrac{20^4 \times 25 \times 8}{20 \times 16}$; $\dfrac{5^6 \times 5^7 \times 5}{5^4 \times 9}$; $\dfrac{16^m \times 14^n}{16^3 \times 14}$.
5. $\dfrac{a^4 \times b}{a \times b}$; $\dfrac{a^m \times b^m}{a \times b^2}$; $\dfrac{p^m \times q^m}{p^n \times q^4}$.

Questions on Multiplication and Division.

Define Multiplication, multiplicand, multiplier, product, and factor. What is a square? How can Multiplication be said to be abridged Addition? What is a distributive operation? Express the product of 7 and 9 in four different ways, showing of what parts the whole is made up.

State the first axiom assumed in Multiplication. Give an example. State the second, and give an example. What rule may be deduced from the second axiom? If in multiplying by 7000 we multiply by 7 and add three ciphers, what principles are illustrated by the process?

What is Long Multiplication? State the rule and the truth on which the rule depends. What method is always employed in Compound Multiplication? Why is no other method allowable? State how a sum may be shortened when 218, 99, 101, or 369 is the multiplier.

Define Reduction. How much of Reduction may be performed by Multiplication? Give an example and state the rule.

What is meant by Division, dividend, quotient, divisor? In how many different ways can you state the problem $a \div b$? What sort of Subtraction sums may be abridged by the Rule of Division? What is the first axiom assumed in Division?

State the Rule for Simple Division. Why is it better to commence at the left hand of the sum? How do you divide by a large number when its factors are known? [illegible] the truth assumed in this rule.

Give the Rule for Long Division. What is meant by partial dividends and quotients? If all the partial dividends be added up, what should be their sum? How does the Rule for Descending Reduction apply to Compound Division? When is the answer to a Division sum really a quotient? When is it not so?

State the methods of proving Multiplication and Division. Show how Ascending Reduction belongs to Division, and why? What is meant by the third or fourth power of a number? Define exponent, and give an example. How does the increase of the multiplier or multiplicand affect the product? State the axioms on this subject.

What change may take place in the factors which will have no effect on the product? Give an example. What inferences may be drawn from the truth you illustrate? Give examples—I. Of three numbers, the product of the first and third equalling the square of the second; and II. Of four numbers, having the product of the first and fourth equal to that of the second and third. State the reason in each case.

Construct a series of numbers such that the first and last shall give the same product as any pair equally distant from them, and give the reason. In Division what is the effect of increasing or diminishing the dividend? the divisor?

What changes can be effected in the divisor and dividend which will not affect the quotient? State the truth in words, and give an example. What practical uses arise out of this principle? Explain cancelling, and give an example. How may different powers of a given number be multiplied or divided?

State in words the truth expressed by each of the following formulæ:—

1. If $x = y + z$, then $ma = my + mz$

2. $xyz = zxy$

3. If $xy = z$, then $mz = (mx)\,y$

4. If $m = p + q$, and $n = y + z$, then $mn = py + pz + qy + qz$

5. If $m = x + y + z$, then $\frac{m}{n} = \frac{x}{n} + \frac{y}{n} + \frac{z}{n}$

6. If $z = xy$, then $\frac{m}{z} = \frac{m \div x}{y}$

7. If $pq = z$, then $p \times mq = mz$, and $p \times \frac{q}{m} = \frac{z}{m}$

8. $pq = mp \times \frac{q}{m}$

9. If $cd = ef$, then $e = \frac{cd}{f}$, and $c = \frac{ef}{d}$

10. If $\frac{m}{n} = o$, then $\frac{xm}{n} = xo$, and $\frac{m \div a}{n} = o + a$

11. If $\frac{m}{n} = o$, then $\frac{m}{pn} = \frac{o}{p}$, and $\frac{m}{n \div p} = op$

12. $\frac{m}{n} = \frac{mx}{nx} = \frac{m \div x}{n \div x}$

13. $a^m \times a^n = a^{m+n}$, and $a^m \div a^n = a^{m-n}$

EXERCISE XLVII.

MISCELLANEOUS EXERCISES ON THE ARITHMETIC OF INTEGERS.

1. The less of two numbers is 347 and their difference 58, what is their product?

2. Find one number equivalent to each of the following expressions:—

$$\frac{289 + 735}{18}; \quad \frac{587 \times 42}{609 - 14}; \quad (237 \times 14 \times 8) \div (19 + 7 - 6).$$

3. What is the price of 1 yard when 48 cost £15 10s. 4d.?

4. Multiply 2s. 4d. by 215; and 3s. 9d. by 175.

5. What will be the amount of the wages of 6 labourers for $28\frac{1}{2}$ days at 2s. 3d. each per day?

6. Divide £3 15s. $5\frac{1}{2}$d. by 23; and £50 by 17.

7. Find the product of the sum and difference of the two numbers, 792 and 68.

8. What is the difference between the sum of the squares of 519 and 432 and the square of their sums?

9. Simplify the following expressions:—

$128 + 19 + 64 - (8 + 123)$.
$796 - 14 + 8 - 3 + 153$.
$2740 \times (58 + 6)$; $3709 \times 695 + 8$.
$2718 - (45 + 6 - 1)$.
$(409 - 8) \times (627 + 14)$; $(5296 + 13) \div (29 - 3)$.

10. How many times will a wheel 7 ft. 3 in. in circumference revolve in traversing 14 miles?

11. How many farthings are there in 27 sovereigns, 50 half-crowns, 87 shillings, and 43 sixpences?

12. How many minutes are there in 365 days, 5 hours, and 48 minutes?

13. In £58 12s. 6d. how many francs at 10d. each?

14. Reduce 125 yds. 2 ft. 4 in. to inches.

15. An English sovereign will exchange in Belgium for 25 francs 20 centimes (100 centimes = 1 franc); in Prussia for 6 thalers 20 groschen (30 groschen = 1 thaler); and in Frankfort for 12 florins (60 kreutzers = 1 florin). How will the sum of £12 14s. 6d. (English) be paid in the money of these countries?

16. Reduce £123 15s. $9\frac{3}{4}$d. to farthings.

17. How many American dollars, value 4s. 2d., are equal in value to £20?

18. How many coins worth 4s. 9d. are there in £231 16s.?

19. How many parcels, each containing $4\frac{1}{4}$ oz., can be made up out of 3 qrs. 17 lbs.?

20. If 17 men reap 19 acres 2 roods 17 poles in a day, and 8 of them reap one-third of an acre each, how much ought each of the others to reap?

21. A silversmith makes 18 spoons, each weighing 2 oz. 7 dwt. 19 grs.; three dozen and a half others weighing 1 oz. 11 dwt. 7 grs.; and 19 silver forks, each weighing 1¾ oz.: how much silver must he use?

22. What is the worth of 17 lbs. 6 oz. 10 dwt. of gold at £3 17s. 10½d. per oz.?

23. If one-thirteenth of a certain gold coinage be alloy, what is the quantity of pure gold in 274 pieces weighing 54 grs. each?

24. An English sovereign weighs 123 grains, what is the weight of £235 10s. in gold?

25. If a man advances 2 ft. 6 in. each step, takes 72 steps per minute, and walks 4½ hours, how far will he go?

26. Find the difference in feet between the polar and the equatorial diameters of the earth, the one being 7,899 miles 1 fur., and the other 7925 miles 5 fur.

27. How much English money is equivalent to 187 Italian ducats, at 3s. 1½d. each?

28. If a plank be 6¾ in. wide, what length of it will give a surface of 2 square feet?

29. How many pieces worth 2½d. each are there in 150 guineas, £70, 34 crowns, 17 half-crowns, and 89 sixpences?

30. A ship is worth £3,700, and the cargo is worth 6 times the ship; what is the worth of one-fifteenth of the ship and cargo together?

31. A pack of wool weighing 2 cwt. 1 qr. 19 lbs. costs £15 4s. 10½d., what is its cost per lb.?

32. A gentleman's yearly income is £900 guineas: he spends one-twelfth of his income in charity, and spends on an average £11 5s. per week; what can he save in a year?

33. Three persons purchase a ship worth £24,000; the first takes two parts; the second, three; and the third, four parts: what is the value of each man's share?

34. What number subtracted from the second power of 29 will leave the product of 16 and 19?

35. If a steam-vessel reach a port 3,050 miles distant in 4 weeks 3 days, what is her average speed per hour?

36. What is the difference between the daily income of a man whose salary is £250 a year and of one who receives £720 per annum?

37. What is the number, from which, if you take the third power of 35, will give the second power of 79 as the answer?

38. What is the total weight of silver in half a dozen dishes, each weighing 49 oz. 3 dwt. 4 grs.; a dozen plates, each 16 oz. 17 dwt.; and a salver weighing 126 oz. 15 dwt. 18 grs.?

39. By how much must the square of the sum of 12 and 3 be multiplied in order to give the third power of the difference between 108 and 59?

40. What is the total length of 49 pieces of cloth, each measuring 27 yds. 2 qrs. 2 nails?

41. If I purchase 17 hogsheads weighing 14 cwt. each, at £24 per cwt., and am allowed 3 lbs. in every cwt. for waste, at what price must I sell in order to gain 2¼d. per lb.?

42. Suppose in London 1 person dies per annum out of every 44, in a manufacturing town 1 in 41, and in a rural village 1 in 49; how long will it be before there is a total of 53,000 deaths in all three, supposing the population of the first to be 2,250,000, the second 783,000, and the third 2,792?

43. How many acres are contained in three countries, of which the first comprises 723,100 square miles, the second 12,342, and the third 89,704 square miles?

44. What number, divided by the sum of the cubes of 3 and 4, will give as quotient the square of the sums of 7 and 8?

45. If I buy 1,874 yards of cloth, at 4s. 6½d. per yard, and sell at 5s. 3d., what do I gain?

46. If a merchant insures his warehouse when stocked at £17,530, but when empty at £2,935, by how much does he calculate that the worth of his stock exceeds that of the warehouse?

47. If 1,792 persons, 292 carriages, and 17 single horses, pass through a toll-gate in one day, the first paying one-halfpenny, the second 2¼d., and the third 1½d. each, how much money is received?

48. How many guineas, sovereigns, crowns, half-crowns, and shillings, and of each an equal number, are there in £1,548?

49. If I buy 1,453 gallons of spirits at 6s. 7d. per gallon, and after losing a quantity of it by an accident, gain £25 profit by selling the remainder at 8s. 10d. per gallon, how much was lost?

50. If I have to measure a distance of three furlongs with a line three rods and a half long, how many times will the line measure the distance?

51. How many pounds of tea at 5s. 6d. per lb. must be given in exchange for 293 yards of silk at 3s. 4½d. per yard?

52. If 2 cwt. 1 qr. of sugar costing £3 5s. per cwt., and 1½ cwt. at 4½ guineas be mixed together, what is the value of a pound of the mixture?

53. A person buys a hogshead of wine in bond for £36, the duty is 5s. 6d. per gallon, what must it be sold at per dozen to gain £15, six bottles being equal to one gallon?

54. If a person gives £46 10s. for 107 gallons, how much water must he add to it in order to reduce its value to 7s. 9d. per gallon?

PRIME AND COMPOSITE NUMBERS.

SECTION I.—MEASURES AND MULTIPLES.*

138. A number is a *measure* of another, or is said to measure it, when it is contained an exact number of times in that other.

Thus: 5 is a measure of 30, 11 of 55, 12 of 36.

139. A number is called a *multiple* of another when it contains that other an exact number of times.

Thus: 48 is a multiple of 6, 24 of 8, and 56 of 7.

140. *Observation.*—"Measure" and "multiple" are correlative terms, as they describe opposite and yet corresponding relationship. Whenever there is a measure, the number of which it is a measure is its multiple, and whenever there is a multiple, the number of which it is a multiple is its measure.

For, because 27 is a multiple of 9, 9 is a measure of 27.
So, if x is a measure of y, y is a multiple of x.

141. A number is called a *common measure*† of two or more others when it is a measure of each of them.

Thus: 7 is a common measure of 21, of 70, and of 14.
If a is a measure of b, c, and d, it is their common measure.

142. A number is called a *common multiple* of two or more others when it is a multiple of each of them.

Thus, because 48 contains an exact number of sixes, of twelves, of eights, it is a common multiple of 6, 8, and 12.

* These words, measure and multiple, are relative terms; they always show the relationship between one number and another, and do not describe any abstract property of either. Thus we cannot simply say 12 is a multiple; but 12 is a multiple *of three* or *of four*. So it would be unmeaning to say that 9 was a measure merely, unless we said it was a measure *of* some other number, as, that 9 is a measure of 27 or of 108.

† The word *common* can never be properly used except when a number is considered in relation to two or more others. It is never right to say that a is a common measure or a common multiple of b, for instance. If it be common it must be common *to two* or more others, as, a is a common measure of b and c.

Observation.—The product of any two numbers is evidently their common multiple, although, as will afterwards be seen (163), it is not the only, and it may not be the least, common multiple.

Thus, 5×6 must evidently contain an exact number both of sixes and of fives, and is therefore their common multiple.

143. Every number is either Prime or Composite.*

A prime number is one which has no measure. All other numbers are composite.

Thus: 5, 7, 11, 19, are prime numbers.

144. Numbers are said to be prime to one another when they have no common measure.

Thus, the number 20 is not a prime number absolutely or considered by itself, nor is the number 9, because both have measures; but as neither of the measures of the one is also a measure of the other, they are *relatively* prime or are prime to one another. Such numbers are sometimes called incommensurable.

145. *Observation.*—The number of measures a given number may have is limited: it may have none at all, or it may have only one or two, and in all cases the number of measures can easily be determined; but the number of multiples a number may have is unlimited, for it is manifest that we may add it to itself an infinite number of times, and that each time we do this we have a new multiple of it. So also if we take two numbers at random; it may be that they have no common measure, and if they have, the number of such measures will probably be very small and can soon be determined, but they will certainly have a common multiple, and there is no limit to the number of these common multiples which may be taken.

146. Axiom XVI.—*If one number measures another it measures all multiples of that other.*

Demonstrative Example.—For if 7 measures 28, or is contained a certain number of times in 28, it must also measure twice 28, or three times 28, or any number of 28's.

General Formula.—If a measures b it must also measure xb.

For let it be granted that a is contained n times in b, then $b = na$; then $xb = xna$ and contains a xn times. But if a number is contained xn times in another it measures that other.

* Prime, from *primus*, first; composite, from *compono*, *compositus*, put together.

147. Axiom XVII.—*If one number measures two others it must also measure their sum.*

Demonstrative Example.—Because 6 is a measure of 12 and also of 18, it must also measure 12 + 18, or 30.

General Formula.—If a measures b and c it measures $b + c$.

For if it measures b it is contained an exact number of times in it; let it be contained n times, then $an = b$. Similarly, let c contain a, m times, then $c = am$; but because $b = an$ and $c = am$, therefore (71) $b + c = a \times (m + n)$; or $b + c$ contains a, $(m + n)$ times, and is therefore a multiple of a.

148. Axiom XVIII.—*If one number measures two others it measures their difference.*

Demonstrative Example.—Because 5 measures 60 and 15 it must also measure 60 − 15, or 45.

General Formula.—If a measures b and c it measures $b - c$ or $c - b$.

As in the last example, let b contain a, n times, and c, m times, then $b - c$ must contain a, $(n - m)$ times, and must therefore be a multiple of a.

149. *If one number measures the divisor and dividend in a Division sum it must also measure the remainder.*

Demonstrative Example.—In the sum, because 6 is a measure both of 24 and of 372, the divisor and dividend, it must also measure the remainder 12; for by (146), because 6 measures 24 (hyp.), it also measures 360, which is a multiple of 24; but 6 also measures 372 (hyp.), and 12 is the difference between 360 and 372. Wherefore (148) 6 also measures this difference.

$$\begin{array}{r} 24)\,372\,(15 \\ 24 \\ \hline 132 \\ 120 \\ \hline 12 \end{array}$$

General Formula.—In the sum, let n be a measure of a and also of b; it must also measure d; for by (146), because it measures a it must measure ca, which is a multiple of a; and it measures b by hypothesis, therefore (148) it must measure d, which is the difference between b and ca.

$$\begin{array}{r} a)\,b\,(c \\ ca \\ \hline d \end{array}$$

150. *If one number measures the divisor and remainder it must also measure the dividend.*

Demonstrative Example.—In the last case it might be inferred from the fact that 6 measures 12 and 24 that it must also measure 372, for this is the sum of 12 and of a multiple of 24.

General Formula.—If n measures both a and d it is also a measure of b.

$$\begin{array}{r} a)b(c \\ ca \\ \hline d \end{array}$$

For n being a measure of a is (146) also a measure of ca, but by hypothesis it also measures d, therefore (147) it must measure the sum of ca and d. But b is the sum of ca and d. Wherefore n measures b.

151. *Corollary.*—From the last two propositions it may be inferred, that *the greatest number which measures the divisor and remainder is also the greatest which will measure the divisor and dividend.* For, because every number which measures the divisor and remainder (150) also measures the dividend, and because every number which measures divisor and dividend (149) also measures remainder, the greatest which is common to divisor and remainder is also the greatest common to dividend and divisor.

152. Greatest Common Measure.—It is often required to find the greatest number which will divide two or more others without a remainder,—in other words to find the *greatest common measure* of those numbers. Whenever the numbers are small, or are within the range of the tables which we know by heart, we may find their greatest measure by simple inspection and without trouble. But whenever the numbers are large we adopt the principle just enunciated, and by making one of the given numbers the divisor, and the other the dividend, we solve the question in the following manner:—

$$\begin{array}{r} 252)\,2097\,(8 \\ 2016 \\ \hline 81)\,252\,(3 \\ 243 \\ \hline 9)\,81\,(9 \\ 81 \\ \hline \end{array}$$

Let it be required to find the greatest common measure of 252 and 2097. These numbers are too large to be determined by simple inspection, so we divide one by another. Now by (151) we know that the greatest common measure of 2097 and 252, the divisor and dividend, will also be the greatest of 252 and 81, the divisor and remainder. But these are also too large. We make one of them the dividend; and again the greatest common measure of this divisor (81) and the new dividend (252) must also be the greatest common measure of 81 and

of 9, the divisor and new remainder. But on applying one of these to the other we find that it is a measure of it, and must therefore be the greatest common measure of both. 9 is therefore the greatest common measure of 252 and 2097.

153. Let it be required to find the greatest common measure of any two numbers, m and n, and let n be the greater. Let it contain m 3 times and leave a remainder o. Let o be contained in m 5 times and leave a remainder p, &c., &c.; and let s the last remainder be contained exactly twice in r. *Then s is a common measure of m and n.*

$$\begin{array}{llllll}
m)\,n\,(3 & & & & & \\
\,3m & & & & & \\
\,\overline{} & o)\,m\,(5 & & & & \\
 & \,5o & & & & \\
 & \,\overline{} & p)\,o\,(8 & & & \\
 & & \,8p & & & \\
 & & \,\overline{} & q)\,p\,(2 & & \\
 & & & \,2q & & \\
 & & & \,\overline{} & r)\,q\,(1 & \\
 & & & & \,r & \\
 & & & & \,\overline{} & s)\,r\,(2 \\
 & & & & & \,2s \\
 & & & & & \,\overline{}
\end{array}$$

For since it measures r it is a common measure of itself and r. But because (150) whatever measures remainder and divisor measures also the dividend, therefore s measures q. Again, for the same reason, since it measures q and r it also measures p, and because it measures p and q (the divisor and remainder) it must also measure o (the dividend). But whatever measures o and p must also measure m, and whatever measures m and o must measure n. But s measures m and o, therefore it measures m and n.

S is also the GREATEST *common measure of m and n.*

For, if it be possible, let them have a greater, and let it be a. Then because, according to this hypothesis, a measures m and n, it also measures o (149). And for the same reason, measuring m and o, the divisor and dividend, a must also measure p, the remainder, and if so must also measure q; and because it measures p and q it also measures the remainder r, and therefore also measures s. But by hypothesis a is greater than s, therefore it cannot measure s, and therefore no greater number than s is a common measure of m and n.*

154. *Observation.*—The object of this process is gradually to diminish the numbers under inspection until they are small enough to have their measure easily ascertained by the tables. We start with two numbers which are too large to allow us to determine their greatest common measure by simple inspection. But as the

* Euclid, book vii. prop. 2.

remainder in a Division sum must always be less than either divisor or dividend, and as whatever is the greatest common measure of the two original numbers is also the greatest common measure of divisor and remainder, the effect of the first step is to give us two smaller numbers to compare. If these two (the divisor and remainder) are not small enough, we treat them in the same way and find another remainder, which is of course smaller than either. Thus we proceed until we obtain two numbers so small that we can readily tell whether they have or have not a common divisor. If they have, that number is also the common divisor of the two original numbers; if they have not, the two original numbers have not any common measure.

If in the course of the operation we perceive that a remainder, being a prime number, does not measure the preceding remainder, we may at once conclude that the two given numbers are prime to each other; or,

If any two consecutive remainders are observed to have no common measure, it is useless to proceed further, because in that case it will be evident that the numbers themselves have no common measure.

RULE TO FIND THE GREATEST COMMON MEASURE OF TWO NUMBERS.

155. Divide the greater by the less. If there be no remainder the less of the two numbers is a measure of the greater and therefore the greatest common measure.

But if there be a remainder, bring down the former divisor and divide it by the first remainder. Afterwards bring down the second divisor and divide it by the second remainder, and so on until there is no remainder. The last divisor is the greatest common measure of the two original numbers.

If the last divisor be 1, the numbers have no common measure, *i.e.*, they are incommensurable.

EXERCISE XLVIII.

Find the greatest common measure of the following numbers:—

1. 35 and 40; 288 and 36; 570 and 930.
2. 266 and 637; 1793 and 462; 125 and 360.

3. 472 and 720; 162 and 2763; 300 and 468.
4. 65935 and 47355; 7228 and 4196; 1286 and 907.
5. 10987 and 1495; 1271 and 31628; 4096 and 84.
6. 4058 and 432; 323 and 1700; 17962 and 815.

156. Suppose it is required to find the greatest common measure of 30, 18, and 21. By (155) we find that the greatest common measure of 30 and 18 is 6, or that whatever measures 30 and 18 must also measure 6. Wherefore, whatever measures 30 and 18 and 21 must also measure 6 and 21; and 3, which is the greatest common measure of 6 and 21, &c., must be the greatest common measure of 30, 18, and 21. It is evident that if any two of the given numbers have no common divisor, the whole of the given numbers have no common divisor, and are therefore incommensurable.

To find the greatest common measure of a, b, c, and d. Let the greatest common measure of a and b be m; then, because whatever measures a and b also measures m, the greatest common measure of a, b, and c must be that of m and c; let this be n; then because whatever measures a, b, and c also measures n, the greatest common measure of a, b, c, and d must be that of n and d; let this be p; then p is the greatest common measure of a, b, c, and d.

RULE TO FIND THE GREATEST COMMON MEASURE OF THREE OR MORE NUMBERS.

157. Find by (155) the greatest common divisor of any two of the given numbers. Then find the greatest common measure of the divisor thus obtained, and another of the given numbers; proceed in this way until the numbers are exhausted. The last of these common divisors will be the number sought.

EXERCISE XLIX.

Find the greatest common measure of the following numbers:—

1. 25, 75, and 100; 24, 16, and 80; 64, 48, and 120.
2. 805, 2622, and 1978; 6914, 396, and 5784; 170, 262, 568.
3. 63, 700, and 371; 108, 6144, and 1116; 729, 1371, and 1695.

158. Axiom XIX.—*Every number is either a prime number or may be resolved into prime factors.*

Demonstrative Example.—For if the number can be divided by another it is not a prime number, and is therefore measured by the divisor and the quotient, which are its factors. If either of these factors has a measure it also may be resolved into its factors, and this process may evidently be carried on until all the factors are prime numbers.

Let a be a number: if it has no measure it is a prime number; but if it has, let it equal $b \times c$; then if b and c are prime numbers a is resolved into its prime factors; if not, let $b = e \times f$ and $c = g \times h$, and let e, f, g, h be prime numbers. Then $a = e \times f \times g \times h$ and is resolved into its prime factors.

RULE TO RESOLVE A NUMBER INTO ITS PRIME FACTORS.

159. Divide by the smallest prime number which will measure it. Then divide the quotient so found by the smallest prime number which it contains; and proceed in this way until a quotient occurs which cannot be divided. The series of divisors and the last quotient are the prime factors.

Example.—Resolve 390 into its prime factors. Here we divide first by 2, then by 3, then by 5, and at last come to a quotient which is a prime number. The number is thus found to consist of $2 \times 3 \times 5 \times 13$. But by the rule, each of the numbers thus chosen was a prime number. Wherefore 390 has been resolved into its prime factors.

```
2)390
 ----
3)195
 ----
 5)65
  ---
   13
```

Ans. $390 = 2 \times 3 \times 5 \times 13$.

Exercise L.

Resolve each of the following numbers into its prime factors:—

1. 347; 58; 196.
2. 1027; 3526; 4098.
3. 7189; 6541; 4127.
4. 23169; 5481; 71086.

160. It is not difficult to ascertain how many prime numbers can be found within any given limit. If we set down a column of all the numbers in order, from 1 to 100, we may mark off those which are not prime numbers in the following manner:—

Begin by marking off every number which contains two, that being the lowest prime number; thus 4, 6, 8, and the whole series of even numbers will be marked. It is evident that not one of these is a prime number, as every one is divisible by 2. Then commence from 3, which is the next prime number, marking off every number which contains 3. Thus 6, 9, 12, 15 and the whole series of multiples of 3 will be excluded from the list of prime numbers. In like manner commence from 5 (the prime number next in order) and point off every number in the series which contains an exact number of fives. Thus 10, 15, 20, &c., will be excluded from the list. Again, take 7 (the next prime number) and mark off its multiples 14, 21, 28, &c.; then do the same with 11, and so on with all the prime numbers in succession. Every number which is not prime will be excluded. In the list, the number of marks affixed to each figure shows the number of its prime factors. In the series of numbers given it will be seen that the only prime numbers are—2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.*

1		14	I I	27	I	39	I I
2		15	I I	28	I I	40	I I
3		16	I	29		41	
4	I	17		30	I I I	42	I I I
5		18	I I	31		43	
6	I I	19		32	I	44	I I
7		20	I I	33	I I	45	I I
8	I	21	I I	34	I I	46	I I
9	I	22	I I	35	I I	47	
10	I I	23		36	I I	48	I I
11		24	I I	37		49	I
12	I I	25	I	38	I I	50	I I
13		26	I I				

* This process of eliminating all the composite numbers from a series was invented by Eratosthenes, a Cyrenian Greek, who had charge of the Alexandrian Library in the time of the second Ptolemy, and paid great attention to Mathematical Science. It is called the sieve of Eratosthenes.

Exercise LI.

Make out a similar list to that in the Example (from 1 to 200), and ascertain how many prime numbers there are.

161. Least Common Multiple.—The smallest number which can be divided by two or more others without a remainder is called the least common multiple of those others. The rule for finding this number depends on several simple considerations.

162. *The product of two or more numbers is a common multiple of all of them.*

Demonstrative Example.—The product of 4, of 7, and 9, must contain an exact number of fours, of sevens, and of nines, and is therefore a multiple of each of them. A common multiple of any set of numbers can always therefore be found by multiplying them together.

General Formula.—If $a \times b = c$, c is a common multiple of a and b.

163. *If two or more numbers have a common measure, their product divided by that common measure will be a common multiple of those numbers.*

Demonstrative Example.—If it be required to find a common multiple of 30 and 12, we may do it at once (162) by taking their product, for $30 \times 12 = 360 =$ a common multiple of 30 and 12. But because 6 is their common measure, and $30 = 6 \times 5$, and $12 = 6 \times 2$, it is evident that $6 \times 2 \times 5$ will be a multiple of both, for it will contain the 12 five times, and the 30 two times. Therefore 60, or 12×30 divided by their greatest common measure, is their least common multiple.

General Formula.—If $a = mx$ and $b = my$, then mxy is a common multiple of a and b.

For mxy contains a y times, and it also contains b x times. It is therefore their common multiple. But since $a = mx$ and $b = my$, $ab = mx \times my$, and $\frac{mxmy}{m} = mxy$, or the product of the two numbers divided by their greatest common measure gives the least common multiple.

RULE TO FIND THE LEAST COMMON MULTIPLE OF TWO NUMBERS.

164. Find (155) their greatest common measure and divide their product by this number.

Exercise LII.

Find the least common multiple of the following numbers :—

1. 25 and 35 ; 72 and 108 ; 63 and 99.
2. 54 and 842 ; 300 and 42 ; 75 and 100.
3. 82 and 96 ; 7123 and 456 ; 372 and 48.
4. 235 and 195 ; 124 and 16 ; 50 and 30.

165. *If in any set of numbers two or more are found having a common factor, the product of these numbers divided by the common factors will be a common multiple of all of them.*

Demonstrative Example.—Suppose it is required to find the least common multiples of the four numbers, 16, 20, 21, and 14. Here, because $16 = 4 \times 4$, $20 = 5 \times 4$, $21 = 3 \times 7$, $14 = 2 \times 7$; therefore $4 \times 4 \times 5 \times 7 \times 3$ will be a multiple of each of them, for it will contain all the factors of each; but this will equal the whole product divided by 4×7.

General Formula.—If $a = mx$, $b = xy$, $c = yz$, and $d = mz$, then $mxyz$ is a common multiple of a, b, c, and d; for it contains a yz times, b mz times, c mx times, and d xy times;

$$\text{and } mxyz = \frac{abcd}{xz}.$$

166. The exclusion of common factors is usually effected by the following process :—

Let it be required to find the least common multiple of 15, 18, 20, 32, 12, and 100.

3)	15	18	20	32	12	100
4)	5	6	20	32	4	100
5)	5	6	5	8	1	25
2)	1	6	1	8	1	5
		3	1	4	1	5

$3 \times 4 \times 5 \times 2 \times 3 \times 4 \times 5 = 7200$ the least common multiple.

Here it was first observed that 3 was a common measure of several of the given numbers, viz., 15, 18, and 12. Dividing each of these by their common factor, we find that (165) $3 \times 5 \times 6 \times 4$ will contain each of them. The other numbers, 20, 32, and 100, were brought down.

At the second line we have therefore this result:—

The product of $3 \times 5 \times 6 \times 20 \times 32 \times 4 \times 100$ is a multiple each of the original numbers.

But of these numbers, three, viz., 20, 32, and 100, are divisible by 4.

Wherefore $4 \times 5 \times 8 \times 25$ is a common multiple of $20 \times 32 \times 100$.

At the third line we have this result:—

$3 \times 4 \times 5 \times 5 \times 6 \times 5 \times 8 \times 25$ is a multiple of each of the figures in the second line, and therefore of each of the original numbers.

Again, dividing three of these numbers by 5 and afterwards by 2 we have

The product of 3, 4, 5, 2, 3, 4, and 5, or 7200, is a common multiple of all the original numbers. This is evidently a much smaller number than the product of 15, 18, 20, 32, 12, and 100.

167. *Observation.*—The division by common factors in any order will always give a *less* common multiple than the product of the given numbers, but it does not always give their *least* common multiple. In order to obtain this it is necessary only to divide by prime numbers, or a number which, like 4, is the square of a prime number.

168. The same result might have been obtained by resolving the numbers into their prime factors by (159). Thus:

$$\begin{aligned} 15 &= 3 \times 5 \\ 18 &= 3 \times 3 \times 2 \\ 20 &= 2 \times 2 \times 5 \\ 32 &= 2 \times 2 \times 2 \times 2 \times 2 \\ 12 &= 3 \times 2 \times 2 \\ 100 &= 5 \times 5 \times 2 \times 2 \end{aligned}$$

Now because the highest power of 2 occurring in any one of these numbers is the 5th (in 32), therefore if we take the fifth power of 2, or $2 \times 2 \times 2 \times 2 \times 2$, in our common multiple all the other twos may be neglected; and because the highest power of 3 which occurs is the 2nd (in 18), therefore if we take 3^2 in the common multiple the other threes may be neglected. Similarly, because 5×5 occurs in the last number

(100) the fives which occur in the other numbers may be neglected. Hence,

$2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5$ or $2^5 \times 3^2 \times 5^2$ or 7200 = the least common multiple of 15, 20, 18, 32, 12, and 100, for it contains all the prime factors of each of them.

But the product of all the numbers is $2^{12} \times 3^4 \times 5^4$, which is a much larger number than 7200.

TO FIND THE LEAST COMMON MULTIPLE OF THREE OR MORE NUMBERS—RULE I.

169. Divide as many of them as possible by any prime number which is a common measure of any of them. Bring down the undivided numbers, and divide those in the second line by any other prime measure of any of them. Proceed in this way until no common measure can be found. The product of all the divisors and of the numbers remaining in the lowest line will be the least common multiple required.

RULE II.

170. Resolve all the given numbers into their prime factors. Take each of these factors to the highest power which occurs in either of the given numbers. The product will be the least common multiple.

Observation.—It is evident that if any one of the numbers is seen to be contained in another it may be struck out at once. Thus, in the example given, because 20 is a measure of 100, whatever is a multiple of 100 will certainly be a multiple of 20, and therefore if a multiple of the 100 be found the 20 may be neglected.

EXERCISE LIII.

Find the least common multiple of the following numbers:—

(a). The first four sums by the second method, and

(b). The whole by the first.

1. 25, 75, and 30; 12, 28, and 44; 29, 16, and 40.
2. 15, 14, 16, and 18; 27, 30, 54, and 27.
3. 105, 110, 14, and 17; 354, 63, 852, and 81.
4. 1, 2, 3, 4, 5, 6, 7, 8, and 9; 20, 12, 15, 18, 4.
5. 2712, 816, 54, and 15; 21, 27, 36, and 19.
6. 5908, 5612, and 3047; 250, 360, 49, and 700.

SECTION II.—SPECIAL PROPERTIES OF NUMBERS.

For convenience in working subsequent rules, and especially in determining the common factors of numbers by simple inspection, the following truths are important.

171. *Every even number is divisible by* 2, *and every number whose two last digits are divisible by* 4 *is itself divisible by* 4.

The former assertion is self-evident; and because $100 = 25 \times 4$ therefore any number of hundreds is a multiple of 4; and if the two last digits also form a multiple of 4, the whole number is divisible by that number, *e.g.*, 1800, 16924, 24936, are divisible by 4.

172. *Every number whose three last digits are divisible by* 8 *is itself divisible by* 8.

For because 1000 is divisible by 8 any number of thousands is so likewise. Hence, if the three last digits form a multiple of 8 the whole is a multiple of 8, *e.g.*, 17000, 25328, and 491720 are divisible by 8.

173. *Every number ending in* 0 *or* 5 *is divisible by* 5.

For every number ending in 0 consists of an exact number of tens, and $10 = 5 \times 2$. Again, every number ending in 5 consists of a number of tens + 5, and is therefore a multiple of 5.

174. *Every prime number terminates with one or other of the digits* 1, 3, 7, *or* 9.

For numbers terminating in any other digit have already been shown to be composite.

175. *Every number the sum of whose digits is divisible by* 9 *or* 3 *is itself divisible by* 9 *or* 3, *and every number the sum of whose digits if divided by* 9 *or* 3 *would give a certain remainder, would itself if divided by* 9 *or* 3 *give the same remainder.*

For because $10 = 9 + 1$, 7 tens must contain 7 nines + 7, 3 tens

= 3 nines + 3, and if any number of tens be taken, it will, if divided by 9, give the same number as a remainder; n tens $\div 9 = n$ times and n remainder.

So also because $100 = 99 + 1$ and $99 = 11 \times 9$, $700 = 7 \times 99 + 7$, and n hundreds divided by 9 will give n remainder. The same is also true of thousands, tens of thousands, millions, &c. Suppose, for example, it is required to divide 723425 by 9.

Now by (93) $\frac{723425}{9} = \frac{700000}{9} + \frac{20000}{9} + \frac{3000}{9} + \frac{400}{9} + \frac{20}{9} + \frac{5}{9}$

If the division be performed separately on each of these parts we shall have a series of remainders, 7, 2, 3, 4, 2, and 5. If the sum of these remainders be divisible by 9 the whole number is divisible by 9, but not otherwise. Here the sum of the digits is 23, or $2 \times 9 + 5$; therefore, if the whole number were divided by 9 the remainder would be 5.

Because 3 is a measure of 9; 10 and every power of 10, if diminished by 1, would be a multiple of 3; therefore 80, 800, 8000, or 800000 if diminished by 8 would have the number 3 for one of its factors. Hence, if the sum of the digits composing a number be divisible by 3 the number itself is divisible by 3.

The method of proving Multiplication and Division by casting out the nines (115) is founded on this truth. For if one factor divided by 9 give a certain remainder, and another factor divided by it give a second remainder, then the product of these two factors divided by 9 will give either the product of these remainders or the same remainder as that product. Hence, if we find by casting out the nines of the multiplicand and multiplier, separately, that the product of these two remainders divided by 9 gives the same remainder as is found by casting out the nines of the answer, that answer is likely to be right; although it is evident that if the answer contained exactly 9 too much or too little, or if a cipher were incorrectly placed in the answer, this method would fail to detect the error.

The method of proving Division will appear exactly the same when it is remembered that the dividend is always the product of the two factors, divisor and quotient.

Observation.—This peculiarity of the number 9 is not an essen-

tial property of the number itself, but is a consequence of our having adopted a decimal notation. Had the base of our system of arithmetic been 5 instead of 10, the same property would have belonged to the number 4; and, generally, if r represent the scale of notation, the sum of the digits divided by $r - 1$ always gives the same remainder as if the whole number were divided by it.*

176. *If an even number have the sum of its digits divisible by* 3 *that number is divisible by* 6.

For because it is even it is divisible by 2, and if it be also divisible by 3 it is divisible by 2×3 or 6. Thus 528, 3738, 2346, are divisible by 6.

177. *If the sum of the digits of an even number be divisible by* 9 *the number is divisible by* 18, *and if the sum be divisible by* 3, *while the tens and units are divisible by* 4, 12 *is a measure of the number.*

The proof of this is the same as in the last case.

178. *Every prime number greater than* 6 *would, if increased or diminished by unity, become divisible by* 6.

For every number greater than 6 would, if divided by 6, leave a remainder 1, 2, 3, 4, 5, or 0. If the remainder be 0 it is not a prime

* Many of the results of the principle here explained are rather curious than useful, although it is a good exercise for pupils to find out similar peculiarities and to trace out their reasons. For example:

If in any line of figures the sum of the digits be subtracted from the whole, the remainder is always a multiple of 9; *and if the figures composing a line be transposed in any order and subtracted from the line itself, the remainder is also divisible by* 9.

Example I.	723425	= a multiple of 9 + 5
Subtract	5	
	723420	= a multiple of 9
II.	723425	= a multiple of 9 + 5
Subtract	23	= the sum of the digits, or $(2 \times 9) + 5$
	723402	= a multiple of 9
III.	723425	= a multiple of 9 + 5
Subtract the same figs. transposed	437522	= a multiple of 9 + 5
	285903	= a multiple of 9

It is evident that having once ascertained that the original number if divided by 9 leaves a remainder 5, we may take away either 5, or 9 + 5, or any number of nines + 5 from that number, and an exact multiple of 9 will be left. But the same figures transposed in any order will always give the same sum; and the number they represent will always therefore be a multiple of 9 + 5.

number. If the remainder be 2 or 4 the number consists of $6n + 2$ or $6n + 4$, and in either case is divisible by 2. If the remainder be 3 the number consists of $6n + 3$ and is divisible by 3. Therefore every prime number is either $6n + 5$ or $6n + 1$. In the former case adding 1 to it, and in the latter case taking 1 from it, would make it a multiple of 6. Hence any one of the prime numbers 7, 11, 13, 17, 19, 23, 29, 31, 37, &c., would, if increased or diminished by unity, become divisible by 6; although it does not follow that all numbers expressible by $6n + 5$ or by $6n + 1$ are prime.

Questions on Prime and Composite Numbers.

Define the words measure, multiple, prime, composite, common measure, common multiple, incommensurable.

Deduce three inferences from the supposition that a measures b and c. State the axioms relating to this subject. What propositions applicable to every Division sum are founded on these axioms?

Give the rule for finding the greatest common measure of two numbers. Take 343 and 539 as an example, and demonstrate each step of the process. How is the greatest common measure of three or more numbers ascertained? Describe the process of resolving a number into prime factors, and of excluding all the composite numbers from a given series.

In what way can a common multiple of two or more numbers always be found? When will a less number serve the purpose? Give the rule for finding the least common multiple—I. of two, and II. of three or more numbers.

How may you tell by simple inspection when a number is divisible by 2, by 4, by 8, or by 5? What numbers are divisible by 9 or by 3, by 18, or by 6? By what test may a prime number be known? Show how the method of proving Multiplication or Division by casting out nines is to be demonstrated.

FRACTIONS.*

SECTION I.—NOTATION OF VULGAR FRACTIONS.

179. That part of Arithmetic which treats of whole numbers is called Integral Arithmetic or the Arithmetic of Integers.† In Integral Arithmetic unity is taken as the standard, and considered capable of *increase*. In Fractional Arithmetic unity is taken as the standard, and considered as capable of *division*. In the former we are concerned only with magnitudes *greater than one;* in the latter with magnitudes *less than one.*‡

The figures 1, 2, 3, 4, 5, &c., all represent different multiples of unity; the expressions which occur in Fractions refer to the *parts* of unity.

180. In (96) and (99) the expressions $\frac{4}{6}$ and $\frac{20}{28}$ were used at the end of Division sums, and stated to mean the *sixth part of four*, and *the twenty-eighth part of twenty*, respectively. We were then not able to do more, with such expressions, than to leave them as divisions which had not been effected. The object of Fractional Arithmetic is to investigate such expressions more fully, and to give us an extended notion of Division.

181. Every fraction represents a quotient, the upper figure being the dividend and the lower the divisor.

182. The lower of the two numbers is called the *denominator* § or namer, because it shows *what parts* of unity have to be taken; and the upper is called the *numerator* § or numberer, because it shows *how many* of these parts are to be taken.

* From *frango*, I break; *fractus*, broken (Latin).

† *Integer* (Latin), whole or undivided. In the word integrity we have it applied to moral character, and signifying singleness of purpose; but in Arithmetic the word always means *unbroken* and the opposite to fractional.

‡ *Preliminary Mental Exercise.*—Before commencing this Rule many questions on the Tables involving both Multiplication and Division should be solved mentally. Such exercises may take two forms, *e.g.*, I. Find $\frac{3}{7}$ of 35, $\frac{7}{12}$ of 96, $\frac{3}{8}$ of 56, $\frac{6}{11}$ of 55, &c., and II. What is that number of which 20 is $\frac{5}{7}$; of which 15 is $\frac{3}{8}$; of which 24 is $\frac{12}{13}$; of which 18 is $\frac{9}{11}$? &c., &c.

§ (Latin) *numero*, I number or count; and *denomino*, I give a name to.

Thus $\frac{8}{9}$ signifies that unity is divided into nine parts, of which eight are taken; it also means the ninth part of eight, or the part which eight is of nine, or the quotient which would be obtained on dividing eight by nine.

183. Fractions are called *proper* when their denominator is greater than their numerator, as $\frac{8}{13}$; and *improper* when the numerator is the greater, as $\frac{9}{8}$, or when both are equal, as $\frac{10}{10}$.

The last expression, $\frac{10}{10}$, means that 1 is divided into 10 equal parts, and that 10 of them are to be taken; in other words $\frac{10}{10}$ means the whole unit; $\frac{11}{10}$ would mean something more than the whole unit.

Whenever the numerator exceeds the denominator the fraction represents more than one, and as this is not a part of unity merely, but a whole number and a part, the expression is called an improper fraction. Whenever the numerator equals the denominator the expression represents exactly one, and is still called *improper* because it is not a fraction at all, but a whole number. But whenever the numerator is less than the denominator it signifies that all the parts into which unity has been divided are not to be taken, and therefore that the expression is really fractional and only represents a part. It is hence called a *proper* fraction.

Nevertheless such expressions as $\frac{11}{10}$ or $\frac{8}{8}$ will often be met with in fractions, and are subject in all respects to the same rules and the same considerations as what are called proper fractions. The distinction therefore is not of much importance, and makes no new rules or statements necessary.

184. A *mixed* number is one which consists of one or more whole numbers and a fraction, as $2\frac{3}{4}$, $5\frac{1}{2}$, $18\frac{7}{13}$, &c.

185. *Every mixed number may have its form altered to that of an improper fraction, and every improper fraction to that of a mixed number.*

Demonstrative Example.—The mixed number $5\frac{7}{8}$ consists of five whole numbers and seven eighths. But because $1 = \frac{8}{8}$ therefore 5 must equal $5 \times \frac{8}{8}$ or $\frac{40}{8}$. Hence $5\frac{7}{8} = \frac{40}{8} + \frac{7}{8}$ or $\frac{47}{8}$.

General Formula.—$a + \frac{b}{c} = \frac{ac}{c} + \frac{b}{c} = \frac{ac + b}{c}$.

186. TO REDUCE A MIXED NUMBER TO AN IMPROPER FRACTION—

RULE.

Multiply the whole number by the denominator of the fraction; add the numerator, and place the denominator under the sum.

EXERCISE LIV.

Reduce the following mixed numbers to improper fractions:—

1. $7\frac{5}{12}$; $8\frac{4}{9}$; $3\frac{7}{11}$.
2. $31\frac{4}{7}$; $54\frac{7}{15}$; $23\frac{14}{19}$.
3. $123\frac{4}{9}$; $21\frac{3}{13}$; $1\frac{6}{7}$.
4. $47\frac{1}{8}$; $2\frac{4}{19}$; $183\frac{13}{27}$.
5. $21\frac{3}{7}$; $15\frac{4}{9}$; $m + \frac{a}{n}$.

187. TO REDUCE AN IMPROPER FRACTION TO A MIXED NUMBER—

RULE.

Divide the numerator by the denominator; the quotient will be a whole number, and the remainder will be the numerator of the fraction.

Example.—Reduce $\frac{37}{5}$ to a mixed number. Here because $\frac{5}{5} = 1$, $\frac{37}{5}$ contains 1 seven times and leaves a remainder 2. There are therefore seven whole numbers and two fifths in $\frac{37}{5}$. That is, $\frac{37}{5} = 7\frac{2}{5}$.

EXERCISE LV.

Reduce the following fractions to mixed numbers:—

1. $\frac{41}{9}$; $\frac{39}{8}$; $\frac{271}{15}$.
2. $\frac{38}{5}$; $\frac{412}{11}$; $\frac{3179}{125}$.
3. $\frac{475}{4}$; $\frac{219}{18}$; $\frac{624}{5}$.
4. $\frac{2105}{17}$; $\frac{3806}{218}$; $\frac{4172}{187}$.
5. $\frac{4198}{365}$; $\frac{6107}{341}$; $\frac{4093}{18}$.
6. $\frac{7296}{81}$; $\frac{312}{75}$; $\frac{4392}{82}$.

188. Every principle which has been demonstrated concerning the relations of dividend, divisor, and quotient (130 *et seq.*), can also be asserted of numerator, denominator, and fraction, *e.g.*—

189. *If the numerator of a fraction be increased or diminished the fraction is increased or diminished in the same degree.* Thus $\frac{6}{7}$ has a certain meaning, but $\frac{12}{7}$ or $\frac{2 \times 6}{7}$ means twice as much, *and* $\frac{3}{7}$ *or* $\frac{6 \div 2}{7}$ *means* only half as much.

190. *If the denominator of a fraction be increased the fraction itself is diminished, but if the denominator be diminished the fraction is increased in the same degree.*

For $\frac{6}{8}$ has a certain meaning, but $\frac{6}{16}$ or $\frac{6}{8 \times 2}$ means only half as much, because here the dividend remains the same and the divisor is doubled, wherefore (131) the quotient is diminished one-half. It is evident that if anything be divided into 8 parts, and again into 16, the sixteenth is only half as much as the eighth, wherefore $\frac{6}{16}$ is half of $\frac{6}{8}$. And by the same reasoning $\frac{6}{4}$ or $\frac{6}{8 \div 2}$ is twice as great as $\frac{6}{8}$.

191. *Corollary I.—A fraction may be multiplied by a whole number either by multiplying its numerator or by dividing its denominator by that number.*

Example.—$\frac{3}{10} \times 2 = \frac{2 \times 3}{10} = \frac{6}{10} = \frac{3}{10 \div 2} = \frac{3}{5}$

General Formula.—$\frac{a}{b} \times c = \frac{ac}{b} = \frac{a}{b \div c}$.

192. *Corollary II.—A fraction may be divided by a whole number either by dividing the numerator or by multiplying the denominator by that number.*

Example.—$\frac{10}{15} \div 5 = \frac{10 \div 5}{15} = \frac{2}{15} = \frac{10}{15 \times 5} = \frac{10}{75}$.

General Formula.—$\frac{a}{b} \div c = \frac{a \div c}{b} = \frac{a}{bc}$.

193. *Observation.*—The second method can only be employed conveniently when the multiplier or divisor is a measure of the denominator or the numerator. In such cases it is preferable to the other.

Exercise LVI.

(a). 1. Multiply $\frac{3}{7}$ by 6; $\frac{2}{9}$ by 3; $\frac{7}{12}$ by 4.

2. $\frac{4}{7} \times 6$; $\frac{3}{19} \times 7$; $\frac{8}{10} \times 5$.

3. $\frac{7}{19} \times 4$; $\frac{3}{5} \times 5$; $\frac{47}{50} \times 2$.

4. $\frac{16}{25} \times 5$; $\frac{8}{14} \times 7$; $\frac{13}{54} \times 17$.

5. $\frac{14}{18} \times 3$; $\frac{9}{16} \times 8$; $\frac{14}{27} \times 9$.

(b). 1. Divide $\frac{7}{15}$ by 6; $\frac{3}{14}$ by 5; $\frac{12}{17}$ by 6.

2. $\frac{7}{12} \div 3$; $\frac{18}{25} \div 6$; $\frac{41}{320} \div 2$.

3. $\frac{15}{19} \div 5$; $\frac{16}{47} \div 8$; $\frac{47}{54} \div 5$.

4. $\frac{21}{69} \div 7$; $\frac{28}{73} \div 14$; $\frac{16}{19} \div 7$.

5. $\frac{12}{73} \div 12$; $\frac{4}{15} \div 3$; $\frac{16}{82} \div 4$

6. $\frac{16}{84} \div 14$; $\frac{27}{80} \div 20$; $\frac{47}{85} \div 10$.

194. *If the numerator and denominator be both multiplied or both divided by the same number, the value of the fraction remains unaltered.**

Demonstrative Example I.—The fraction $\frac{6}{15}$ has a certain meaning. But by (189) if the numerator be multiplied by 2 it becomes $\frac{12}{15}$, or twice as much; while if the denominator be multiplied by 2 the fraction becomes $\frac{6}{30}$, or half as much. But if both be multiplied by 2, and it becomes $\frac{12}{30}$, it is manifest that the one change will neutralize the other, and that the fraction will remain unaltered. Similarly, if both the numerator and denominator of $\frac{6}{15}$ be divided by 3, and it becomes $\frac{2}{5}$, it will have been increased by one process just as much as it is diminished by the other, and will therefore be unaltered.

* It is necessary to notice that equal *additions* to both numerator and denominator *do* make an important difference in the value of the fractions. For example, in the series of fractions—

$$\frac{3}{5},\ \frac{4}{6},\ \frac{5}{7},\ \frac{6}{8},\ \frac{7}{9},\ \frac{8}{10},\ \frac{9}{11},\ \frac{10}{12},\ \frac{11}{13},\ \frac{12}{14},\ \frac{13}{15},\ \frac{14}{16},$$

it may be noticed that *one* is added to both numerator and denominator at each successive step. But because the original numerator, 3, is less than the denominator, 5, the increase is *relatively* greater to the numerator than to the denominator, and the fraction is increased at each step. Nevertheless there is a limit to this increase, for equal additions can never make the fraction mean so much as *one*. $\frac{1000003}{1000005}$ would still be less than unity.

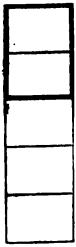
If, however, the fraction be improper, as $\frac{9}{4}$, it is obvious that equal additions will increase the denominator *relatively* in the greatest degree, and that therefore they will have the effect of diminishing the fraction; thus, in the following series of fractions, which are formed by equal additions of one,—

$$\frac{9}{4},\ \frac{10}{5},\ \frac{11}{6},\ \frac{12}{7},\ \frac{13}{8},\ \frac{14}{9},\ \frac{15}{10},\ \frac{16}{11},\ \frac{17}{12},\ \frac{18}{13},\ \frac{19}{14},\ \frac{20}{15},$$

every fraction is less than that on its left. In this case the limit of unity is again constantly approached but never reached, for it is obvious that if a million were added to both numerator and denominator and it became $\frac{1000009}{1000004}$, it would not be so small as one.

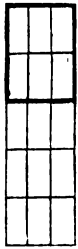
$\frac{a+n}{b+n}$ is therefore greater or less than $\frac{a}{b}$, according as $\frac{a}{b}$ is a proper or improper fraction, while $\frac{a-n}{b-n}$ is less than $\frac{a}{b}$ if that fraction is proper, but greater if it is *improper*.

a



Demonstrative Example II.—In the diagram (*a*) the dark line surrounds a space which is evidently two-fifths ($\frac{2}{5}$) of the whole surface. In (*b*) the dark line surrounds a space which is the same fraction of the whole but contains six fifteenths ($\frac{6}{15}$). For because in *a* each portion is three times as large as each in *b*; three times as many portions are required in the latter to express any given magnitude. Hence $\frac{2}{5} = \frac{6}{15}$.

b



General Formula.—$\frac{a}{b} = \frac{an}{bn} = \frac{a \div n}{b \div n}$.

195. Every fraction may be expressed in an unlimited number of forms. When it is expressed by higher numbers the new numerator and denominator are formed by equal multiplication, and are called *equi-multiples* of the original figures. When it is expressed in a lower name the numerator and denominator are formed from the first by equal division, and are sometimes called *equi-sub-multiples* of the original numbers.

196. By (145) it will be seen that the number of multiples a given number may have is unlimited, but the number of measures is limited. Hence every fraction may be expressed in an infinite number of ways by using two greater numbers; but it can never be expressed by lower numbers unless the numerator and denominator have a common measure.

Exercise LVII.

(a). Express each of the following fractions in four different ways:—

1. $\frac{5}{8}$; $\frac{10}{13}$; $\frac{31}{40}$.
2. $\frac{3}{12}$; $\frac{16}{20}$; $\frac{41}{54}$.
3. $\frac{7}{10}$; $\frac{25}{100}$; $\frac{14}{18}$.
4. $\frac{65}{80}$; $\frac{45}{90}$; $\frac{21}{84}$.

(b). Express each of the following numbers in fractional forms:—

1. 17 with the denominators 3, 8, and 7.
2. 4 .... 5, 9, and 6.

3. 12 with the denominators 8, 3, and 4.
4. 4 .... 15, 12, and 9.
5. 9 .... 21, 3, and 14.

197. For convenience of working it is often necessary to express a given fraction in its *lowest terms.* Whenever the fraction is thus expressed the numerator and denominator must be prime to one another, for otherwise they could be divided by their common measure.

198. To reduce a fraction to its lowest name—

RULE.

Find by (155) the greatest common measure of the numerator and denominator. Divide both by that measure. The resulting fraction will be equal to the first and expressed in its lowest terms.

Example.—Reduce $\frac{252}{1084}$ to its lowest terms.

```
Here by (155) 252) 1084 (4
                   1008
                   ----
                     76) 252 (3
                         228
                         ---
                          24 (76 (3
                              72
                              --
                               4) 24 (6
                                  24
```

4 is found to be the greatest common measure of 252 and 1084. Hence,

$$\frac{252}{1084} = \frac{252 \div 4}{1084 \div 4} = \frac{63}{271}.$$

EXERCISE LVIII.

Reduce the following fractions to their lowest names :—*

1. $\frac{173}{273}$; $\frac{113}{3019}$; $\frac{18}{81}$.
2. $\frac{228}{836}$; $\frac{60}{138}$; $\frac{1100}{8723}$.
3. $\frac{616}{[illegible]}$; $\frac{372}{3600}$; $\frac{784}{791}$.
4. $\frac{330}{684}$; $\frac{1184}{1360}$; $\frac{1001}{1036}$.
5. $\frac{532}{931}$; $\frac{192}{5120}$; $\frac{1428}{2652}$.
6. $\frac{295}{6560}$; $\frac{327}{3465}$; $\frac{396}{1485}$.

* In many cases this may be done by simple inspection. See section on Special *Properties* of Numbers.

199. In order to add, subtract, or in any way compare fractions it is always necessary to reduce them to a common denominator.

If two fractions, $\frac{5}{9}$ and $\frac{3}{7}$, are taken and it is required to find which is the greater, the question is not readily answered. For although by (131) $\frac{1}{9}$ is less than $\frac{1}{7}$, it is not easy to say how much less; nor is it easy to say whether 5 of the ninths may not be greater than 3 of the sevenths. If both had the same denominator, and if the question were, "How much greater is $\frac{5}{9}$ than $\frac{3}{9}$?" the answer would clearly be $\frac{2}{9}$. But in their present form the values of the two fractions cannot be compared.

200. It is the same here as with the Addition and Subtraction of Integers; it was explained in (30) that two concrete numbers could not be added or subtracted unless they referred to the same magnitude. Thus, 5 shillings and 4 pence do not make 9 of either, but if we call the 5 shillings 60 pence, then we may say 60 pence + 4 pence = 64 pence. In this case we could not effect addition until we had found, for the objects to be added, a name which applied to both. In other words we reduced them to a "common denominator," and then the process became one of Simple Addition or Subtraction.

Now the same thing is needed in adding or subtracting fractions. We want to find a name which will apply equally to each of them. It has been shown that all fractions (195) admit of being expressed in an infinite number of forms, by applying the same multiplier to numerator and denominator. It has also been shown (145) that a common multiple may always be found for any set of numbers. Therefore if we choose a common multiple of all the denominators this number will serve as the denominator for each of the fractions. Suppose, for example, we wish to compare together $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$: we observe that 60 is a common multiple of the three denominators; and because 60 is a multiple of 3, any fraction having 3 for its denominator may be so multiplied as to have 60 for its denomitor (196). So also because 60 is a multiple of 4 and of 5, any fractions whatever, which have 4 or 5 for their denominators, may be expressed with the denominator 60. Now no smaller number than 60 would have served the purpose here, because 60 is the least common multiple of the 3 denominators in question.

201. *We may take any multiple whatever of a denominator to serve as the denominator of an equivalent fraction, but when two or more fractions have to be compared it is necessary to take a common multiple of all the denominators.*

Demonstrative Example.—Let it be required to reduce $\frac{2}{3}$, $\frac{4}{5}$, and $\frac{6}{7}$ to a common denominator. We first choose the product of the 3 denominators to serve as the common denominator of all three.

$$\frac{2}{3} = \frac{2 \times 5 \times 7}{3 \times 5 \times 7} = \frac{70}{105}$$

$$\frac{4}{5} = \frac{4 \times 3 \times 7}{5 \times 3 \times 7} = \frac{84}{105}$$

$$\frac{6}{7} = \frac{6 \times 3 \times 5}{7 \times 3 \times 5} = \frac{90}{105}$$

This number is 105. Now, first we want a fraction which shall be equal to $\frac{2}{3}$, and yet which shall have 105 for its denominator. To make 105, the denominator, the 3 has been multiplied by 5×7. But unless (194) whatever is done to the denominator is done to the numerator the fraction will not remain the same. Wherefore we multiply the 2 also by 5×7; and $\frac{2}{3} = \frac{2 \times 5 \times 7}{3 \times 5 \times 7} = \frac{70}{105}$.

Again, in the case of the second fraction, $\frac{4}{5}$, we ask ourselves what has been done to the denominator 5 to make it 105? The answer is, "It has been multiplied by 3×7." We infer, therefore, that the numerator also must be multiplied by 3×7, and hence that $\frac{4}{5} = \frac{4 \times 3 \times 7}{5 \times 3 \times 7} = \frac{84}{105}$.

In the third place we propose the same question, "What has been done to this denominator 7 to make it 105?" It has been multiplied by 3×5. Therefore the numerator 6 must be multiplied in the same manner, and $\frac{6}{7} = \frac{6 \times 3 \times 5}{7 \times 3 \times 5}$ or $\frac{90}{105}$.

202. *Observation.*—When fractions are thus reduced into the same form it is easy to compare them, to say which is the greatest or least, to add them together, or to find the difference between any two of them. Thus, in the example just given, it appears that $\frac{4}{5}$ is the greatest fraction of the three, for it contains 84 "one hundred and fifths," while $\frac{6}{7}$ contains 72, and $\frac{2}{3}$ contains 70 of them. But until they are thus reduced we cannot tell their sum or difference or in any way compare them accurately.

General Formula.—$\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, = \frac{adfh}{bdfh}, \frac{cbfh}{bdfh}, \frac{ebdh}{bdfh}, \frac{gbdf}{bdfh}.$

TO REDUCE FRACTIONS TO A COMMON DENOMINATOR—

RULE.

203. Multiply all the denominators together for the common denominator. Then multiply each numerator by all the denominators except its own, for each of the successive numerators required.

Example.—Reduce $\frac{4}{5}$, $\frac{12}{7}$, $\frac{6}{10}$, and $\frac{3}{4}$, to a common denominator.

$$\frac{4}{5} = \frac{4 \times 7 \times 10 \times 4}{5 \times 7 \times 10 \times 4} = \frac{1120}{1400} \qquad \frac{6}{10} = \frac{6 \times 5 \times 7 \times 4}{10 \times 5 \times 7 \times 4} = \frac{840}{1400}$$

$$\frac{12}{7} = \frac{12 \times 5 \times 10 \times 4}{7 \times 5 \times 10 \times 4} = \frac{2400}{1400} \qquad \frac{3}{4} = \frac{3 \times 5 \times 7 \times 10}{4 \times 5 \times 7 \times 10} = \frac{1050}{1400}$$

EXERCISE LIX.

Reduce the following fractions to a common denominator:—

1. $\frac{2}{7}$, $\frac{3}{8}$, and $\frac{4}{5}$.
2. $\frac{7}{12}$, $\frac{3}{15}$, and $\frac{1}{14}$.
3. $\frac{5}{19}$, $\frac{6}{17}$, and $\frac{3}{20}$.
4. $\frac{4}{7}$, $\frac{15}{16}$, $\frac{3}{8}$, and $\frac{4}{17}$.
5. $\frac{7}{12}$, $\frac{3}{13}$, $\frac{5}{16}$, and $\frac{2}{3}$.
6. $\frac{3}{11}$, $\frac{4}{120}$, $\frac{3}{10}$, and $\frac{7}{15}$.
7. $\frac{2}{13}$, $\frac{6}{28}$, and $\frac{4}{9}$.
8. $\frac{5}{16}$, $\frac{4}{19}$, and $\frac{3}{17}$.
9. $\frac{14}{18}$, $\frac{17}{19}$, and $\frac{21}{22}$.
10. $\frac{13}{17}$, $\frac{12}{21}$, $\frac{3}{100}$, and $\frac{6}{19}$.
11. $\frac{7}{15}$, $\frac{8}{20}$, $\frac{3}{16}$, and $\frac{5}{14}$.
12. $\frac{3}{17}$, $\frac{8}{24}$, $\frac{13}{15}$, and $\frac{4}{50}$.

204. The process just described is applicable to all fractions whatever, but is not always the best, for it is desirable (197) in dealing with fractions to express them by numbers as small as possible, as they can in such cases be more readily dealt with. Therefore whenever the denominators of a fraction have a less common multiple than their product, the least common multiple will be a better common denominator than that which would be obtained by the last rule.

$$\frac{5}{12} = \frac{5 \times 5}{12 \times 5} = \frac{25}{60}$$

$$\frac{3}{15} = \frac{3 \times 4}{15 \times 4} = \frac{12}{60}$$

$$\frac{1}{6} = \frac{1 \times 10}{6 \times 10} = \frac{10}{60}$$

$$\frac{7}{10} = \frac{7 \times 6}{10 \times 6} = \frac{4}{60}$$

For instance, if it be required to reduce $\frac{5}{12}$, $\frac{3}{15}$, $\frac{1}{6}$, and $\frac{7}{10}$, to a common denominator, we may by (169) find that 60 is the least common multiple of all the denominators. Thus, in the first case, because 12 is multiplied by 5 to make 60, the numerator is also multiplied by 5; and because in the fraction $\frac{3}{15}$, 15 is multiplied by 4, the numerator 3 must also be multiplied by 4. So also on dividing 60 by 6, we find that the denominator of the third fraction has been increased 10 times, and must therefore increase the

1, ten times. The last fraction, $\frac{7}{10}$, has its numerator increased 6 times because its denominator has been so increased.

205. Two things only require to be kept in view in this and the former rule.

I. The common denominator chosen must always be a common multiple of all the denominators.

II. Whatever is done to the denominator of each fraction to produce the common denominator must also be done to the numerator of that fraction.

206. TO REDUCE FRACTIONS TO A COMMON DENOMINATOR WHEN THEIR DENOMINATORS HAVE A LESS COMMON MULTIPLE THAN THEIR PRODUCT—

RULE.

> Find the least common multiple of the denominators, divide this multiple by each of the denominators in succession, and multiply each numerator by the quotient thus found.

Example.—Reduce $\frac{7}{18}$, $\frac{5}{20}$, $\frac{4}{5}$, $\frac{7}{12}$, and $\frac{9}{10}$, to a common denominator.

By (106) it may be found that 180 is the least common multiple of the denominators 18, 20, 5, 12, and 10.

$\frac{7}{18} = \frac{7 \times 10}{18 \times 10} = \frac{70}{180}$. $\left\{\begin{array}{c}\text{Here the numerator is}\\ \text{multiplied by 10.}\end{array}\right\}$ Because $180 \div 18 = 10$.

$\frac{5}{20} = \frac{5 \times 9}{20 \times 9} = \frac{45}{180}$. .... 9 .... $180 \div 20 = 9$.

$\frac{4}{5} = \frac{4 \times 36}{5 \times 36} = \frac{144}{180}$. ....36 .... $180 \div 5 = 36$.

$\frac{7}{12} = \frac{7 \times 15}{12 \times 15} = \frac{105}{180}$. ....15 .... $180 \div 12 = 15$.

$\frac{9}{10} = \frac{9 \times 18}{10 \times 18} = \frac{162}{180}$. ....18 .... $180 \div 10 = 18$.

EXERCISE LX.

Reduce the following fractions to a common denominator:—

1. $\frac{3}{15}$ and $\frac{4}{10}$.
2. $\frac{7}{12}$ and $\frac{4}{18}$.
3. $\frac{6}{20}$, $\frac{3}{15}$, and $\frac{4}{7}$.
4. $\frac{5}{9}$, $\frac{3}{21}$, and $\frac{7}{12}$.
5. $\frac{4}{18}$, $\frac{6}{24}$, and $\frac{4}{5}$.
6. $\frac{4}{24}$, $\frac{3}{10}$, and $\frac{15}{60}$.
7. $\frac{2}{15}$, $\frac{3}{75}$, and $\frac{19}{21}$.
8. $\frac{2}{11}$, $\frac{3}{16}$, and $\frac{14}{44}$.
9. $\frac{2}{7}$, $\frac{3}{18}$, $\frac{14}{16}$, and $\frac{12}{28}$.
10. $\frac{2}{12}$, $\frac{14}{24}$, $\frac{7}{15}$, and $\frac{3}{20}$.
11. $\frac{6}{150}$, $\frac{4}{24}$, $\frac{7}{18}$, and $\frac{19}{150}$.
12. $\frac{3}{7}$, $\frac{4}{14}$, and $\frac{12}{2100}$.

SECTION II.—ADDITION AND SUBTRACTION OF VULGAR FRACTIONS.

207. *Whenever fractions have the same denominator they may be added together by adding their numerators only.*

Demonstrative Example.—$\frac{4}{9} + \frac{6}{9} + \frac{1}{9} + \frac{15}{9} = \frac{4 + 6 + 1 + 15}{9} = \frac{26}{9}$.

The denominator is the *namer* of the parts into which the unit is divided; consequently all the fractions which have the same denominator refer to the same parts of unity. Hence $\frac{3}{7}$ and $\frac{2}{7}$ must make $\frac{5}{7}$ or $\frac{3+2}{7}$ just as truly as £3 and £2 make £5.

General Formula.—$\frac{a}{b} + \frac{c}{b} + \frac{x}{b} = \frac{a + c + x}{b}$.

RULE FOR ADDITION OF FRACTIONS.

208. Reduce the fractions to a common denominator by (203), add the numerators only, and place the common denominator under the sum.

209. *Observation.*—Sometimes the numbers to be added together are improper fractions or mixed numbers. If the former, they should be reduced to mixed numbers, and the whole numbers should then be added by themselves, and also the fractions by themselves.

Example.—Find the sum of $5\frac{3}{4}$, $8\frac{2}{10}$, $12\frac{4}{5}$, $2\frac{3}{8}$.

By (9) these may be added in any order, thus:—

$5\frac{3}{4} + 8\frac{2}{10} + 12\frac{4}{5} + 2\frac{3}{8} = 5 + 8 + 12 + 2 + \frac{3}{4} + \frac{2}{10} + \frac{4}{5} + \frac{3}{8}$.

Adding together the whole numbers and bringing the fractions to a common denominator, we have—

$$27 + \frac{30}{40} + \frac{8}{40} + \frac{32}{40} + \frac{15}{40} = 27\frac{85}{40} = 29\frac{5}{40}.$$

EXERCISE LXI.

1. $\frac{2}{3} + \frac{4}{7}$.
2. $\frac{3}{12} + \frac{5}{15} + \frac{4}{17}$.
3. $\frac{2}{7} + \frac{14}{19} + \frac{1}{3}$.
4. $2\frac{7}{12} + 3\frac{4}{5}$.
5. $17\frac{1}{3} + 6\frac{1}{9} + \frac{4}{13}$.
6. $\frac{2}{10} + \frac{12}{55} + \frac{108}{112}$.
7. $4\frac{1}{3} + 6\frac{2}{7} + \frac{13}{29}$.
8. $4\frac{1}{4} + \frac{16}{5}$.
9. $\frac{13}{4} + \frac{17}{3} + \frac{1}{8}$.
10. $\frac{11}{10} + \frac{5}{100} + \frac{39}{1000}$.
11. $\frac{4}{13} + \frac{6}{17} + \frac{3}{11}$.
12. $\frac{17}{12} + 2\frac{1}{5} + \frac{15}{9}$.
13. $\frac{284}{3} + \frac{51}{12} + \frac{6}{7}$.
14. $\frac{13}{8} + 14\frac{5}{8} + \frac{21}{23}$.
15. $\frac{16}{19} + \frac{4}{7} + 100\frac{3}{5}$.
16. $\frac{4}{10} + 10\frac{1}{4} + 3\frac{1}{5}$.
17. $21\frac{2}{14} + \frac{2}{9} + 1\frac{5}{31}$.
18. $12\frac{1}{8} + \frac{21}{4} + \frac{3}{15}$.
19. How much of a ship does a person own who has purchased at different times $\frac{1}{5}$, $\frac{3}{8}$, $\frac{1}{4}$, and $\frac{7}{40}$?

SUBTRACTION OF VULGAR FRACTIONS.

210. *Whenever two fractions have the same denominator, their difference can be found by finding the difference between their numerators.*

Demonstrative Example.—$\frac{12}{17} - \frac{8}{17} = \frac{12 - 8}{17} = \frac{4}{17}$.

General Formula.—$\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$.

211. *Observation.*—If the numbers be mixed numbers it is more convenient first to reduce them to improper fractions, and then perform the subtraction as in the rule.

Example.—$7\frac{2}{3} - 3\frac{4}{5}$. Here it would not be easy, as in Addition, to deal with the whole numbers by themselves, because $\frac{4}{5}$, as such, cannot be taken from $\frac{2}{3}$.

Hence by (185) $7\frac{2}{3} = \frac{23}{3}$, and $3\frac{4}{5} = \frac{19}{5}$.
By (203) $\frac{23}{3} = \frac{115}{15}$, and $\frac{19}{5} = \frac{57}{15}$.
By (210) $\frac{115}{15} - \frac{57}{15} = \frac{115 - 57}{15} = \frac{58}{15}$.
By (187) $\frac{58}{15} = 3\frac{13}{15}$ = the answer.

RULE FOR THE SUBTRACTION OF FRACTIONS.

Reduce them to a common denominator (203); subtract the less numerator from the greater, and place the common denominator underneath this difference.

Exercise LXII.

Find the difference between—

1. $\frac{7}{13}$ and $\frac{3}{12}$.
2. $\frac{24}{32}$ and $\frac{22}{24}$.
3. $\frac{17}{19}$ and $\frac{5}{7}$.
4. $\frac{6}{18}$ and $\frac{3}{4}$.
5. $\frac{7}{10}$ and $\frac{13}{15}$.
6. $\frac{2}{19}$ and $1\frac{7}{12}$.
7. $21\frac{3}{5} - 17\frac{7}{9}$.
8. $1\frac{4}{7} - \frac{17}{18}$.
9. $\frac{27}{4} - \frac{3}{15}$.
10. $14\frac{1}{7} - 3\frac{2}{5}$.
11. $26\frac{2}{5} - 18\frac{3}{13}$.
12. $\frac{103}{12} - \frac{12}{103}$.
13. $172\frac{1}{8} - \frac{97}{4}$.
14. $21\frac{2}{13} - 7\frac{1}{9}$.
15. $\frac{40}{7} + \frac{3}{12} - \frac{5}{16}$.
16. Find the sum and difference of $\frac{2}{3}$ and $\frac{5}{6}$.
17. To what number can I add $7\frac{3}{5}$ so as to make $24\frac{7}{12}$?
18. By how much does the sum of $30\frac{5}{8}$ and $6\frac{2}{7}$ exceed the sum of $10\frac{3}{11}$ and $\frac{179}{18}$?

SECTION III.—MULTIPLICATION OF VULGAR FRACTIONS.

212. Multiplication has been described (56) as a method of increasing a number or taking it a certain number of times. In all cases in which integer numbers are the multipliers this definition is true, and the effect of multiplication is to increase the multiplicand.

For every integer represents a certain number of units, while a fraction always represents certain parts of a unit; to multiply by an integer is, therefore, to take the multiplicand a certain number of times, while to multiply by a fraction is to take the multiplicand certain *parts* of a time. The effect, therefore, of multiplying by a number less than unity is not to increase but to diminish the multiplicand. The definition given of Multiplication in (57) requires to be somewhat expanded in order to meet this case.

213. To multiply one number by another is—

I. To take the multiplicand as many times, *or parts of a time*, as there are units in the multiplier.

II. To find a number which is as many times more *or less* than the multiplicand as the multiplier is more *or less* than unity.

III. To do to the multiplicand whatever has been done to unity to make the multiplier.

From this it appears that to multiply by a fraction, say $\frac{3}{4}$, is to take the given number $\frac{3}{4}$ of a time, that is to take $\frac{3}{4}$ of it. The word *of*, placed between two fractions, means exactly the same as the sign ($\times$) of Multiplication, thus: to multiply $\frac{5}{8}$ by $\frac{11}{12}$ is to take $\frac{11}{12}$ of $\frac{5}{8}$. Reducing a Compound Fraction to a simple one is therefore only a form of Multiplication, and the reasoning and rule which apply to one case apply equally to the other.

214. *We multiply by a fraction when we multiply by its numerator and divide by its denominator.*

215. *Demonstrative Example I.*—Let it be required to multiply 10 by $\frac{6}{7}$ or to take $\frac{6}{7}$ of 10. We first take $\frac{1}{7}$ of 10, or divide 10 by 7; by (192) this is $\frac{10}{7}$; but it was required to take *six* sevenths of it, therefore we multiply this by 6, and $\frac{10}{7} \times 6 = \frac{10 \times 6}{7} = \frac{60}{7}$.

216. *Demonstrative Example II.*—Let it be required to multiply $\frac{4}{5}$ by $\frac{7}{9}$, that is to say (182) to multiply $\frac{4}{5}$ by the ninth part of 7. We

first multiply it by 7. Now to multiply a fraction by a whole number is (191) to multiply its numerator. Wherefore $\frac{4}{5}$ multiplied by 7 equals $\frac{28}{5}$. But it was not required to multiply by 7, but by the ninth part of 7. Wherefore $\frac{28}{5}$ is nine times too great, and the required answer must be one-ninth of this fraction. But (192) to take one-ninth of a fraction is to multiply its denominator by 9, and $\frac{1}{9}$ of $\frac{28}{5} = \frac{28}{9 \times 5}$ or $\frac{28}{45}$. But this answer would have been obtained at once by multiplying by the numerator and dividing by the denominator.

217. *Demonstrative Example III.*—Suppose it be required to multiply $\frac{7}{12}$ by $\frac{3}{8}$. This is equivalent to taking $\frac{3}{8}$ of $\frac{7}{12}$ (213). Let us first take $\frac{1}{8}$ of $\frac{7}{12}$, that is to say, divide $\frac{7}{12}$ by 8. Now by (192) to divide a fraction is to multiply its denominator, whence $\frac{7}{12} \div 8 = \frac{7}{12 \times 8} = \frac{7}{96}$. One eighth of the fraction $\frac{7}{12}$ has now been taken. But it was not required to find $\frac{1}{8}$ but $\frac{3}{8}$; wherefore $\frac{7}{96}$ is 3 times too little: we must therefore multiply it by 3. But (191) to multiply a fraction by 3 we must multiply its numerator, and $\frac{7}{96} \times 3 = \frac{7 \times 3}{96}$ $= \frac{21}{96}$. But this is the answer which would have been found at once by multiplying the numerators together and also the denominators.

218. *Demonstrative Example IV.*—To multiply $\frac{4}{9}$ by $\frac{8}{11}$, we have (213) to do to the multiplicand ($\frac{4}{9}$) whatever has been done to 1 in order to make $\frac{8}{11}$. But to make $\frac{8}{11}$ unity has been multiplied by 8 and divided by 11. We have therefore to multiply $\frac{4}{9}$ by 8 and divide it by 11. But (191) to multiply by 8 is to multiply the numerator, and to divide by 11 (192) is to multiply the denominator. Wherefore $\frac{4}{9} \times \frac{8}{11} = \frac{8 \times 4}{9 \times 11} = \frac{32}{99}$.

General Formula.—$\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ace}{bdf}$ or, $\frac{a}{b}$ of $\frac{c}{d} = \frac{ac}{bd}$.

219. TO MULTIPLY FRACTIONS, OR TO REDUCE A COMPOUND FRACTION INTO A SIMPLE ONE—

RULE.

Multiply the numerators together for a new numerator, and the denominators for a new denominator.

220. *Observation I.*—It generally saves trouble here to bring all mixed numbers into improper fractions before multiplying them, and then to reduce the answer, if an improper fraction, to a mixed number.

Observation II.—Three inferences can easily be deduced from the explanation in (212). I. That when a number is multiplied by a proper fraction, the answer is always as much less than the multiplicand as the numerator of the multiplier is less than its denominator. II. That the product of any two or more proper fractions is less than either of the factors; and III. That the square, cube, &c., of any proper fraction is always less than the fraction itself.

Observation III.—Whenever the same numbers occur in the numerators and denominators of any of the fractions which are to be multiplied or compounded, they may be cancelled or struck out, by (136).

Exercise LXIII.

Solve the following expressions :—

1. $\frac{4}{7} \times \frac{2}{5}$; $\frac{3}{13} \times \frac{12}{17}$; $\frac{5}{9} \times \frac{20}{7} \times 2\frac{1}{4}$.
2. $\frac{6}{13} \times \frac{3}{5} \times \frac{14}{7}$; $\frac{21}{8} \times \frac{14}{8}$; $3\frac{2}{3} \times 5\frac{4}{9}$.
3. $\frac{1}{7}$ of $\frac{3}{8}$ of $\frac{5}{11}$; $\frac{2}{9}$ of $\frac{5}{16}$; $\frac{4}{7} \times \frac{2}{3}$ of $\frac{3}{4}$.
4. $2\frac{1}{4}$ ($\frac{3}{7}$ of $\frac{4}{12}$); $\frac{4}{7} \times 3\frac{1}{2} \times \frac{15}{4}$; $41\frac{7}{10} \times \frac{2}{7} \times \frac{6}{9}$.
5. $(\frac{3}{7} + \frac{5}{8}) \times$ ($\frac{2}{3}$ of $\frac{4}{9}$); $\frac{4}{17}$ of $\frac{2}{9} -$ $\frac{1}{19}$ of $\frac{1}{10}$.
6. $(4\frac{1}{3} + 2\frac{1}{7}) - (1\frac{2}{8} \times \frac{1}{8})$; ($\frac{5}{7}$ of $\frac{1}{9}$) $+ (\frac{2}{3} \times \frac{4}{17})$.
7. $100\frac{3}{5} \times 253\frac{1}{7}$; $(409\frac{3}{8} + 2\frac{2}{9}) \times 15\frac{6}{7}$; $31\frac{4}{17} \times \frac{16}{23}$.
8. $\frac{2}{7}$ of $\frac{3}{4}$ of 100; ($\frac{5}{16}$ of 12) $\times$ ($\frac{3}{11}$ of 7); $\frac{2}{3} \times \frac{7}{8}$ of $\frac{9}{10}$.
9. What is the product of the sum and difference of $\frac{2}{11}$ and $\frac{4}{15}$?
10. How much must be added to $\frac{2}{17}$ of 50 to make $\frac{1}{12}$ of $\frac{2}{3}$ of 1250?
11. From what number can I subtract the product of $\frac{15}{16}$ and $\frac{3}{14}$ so that the remainder may be the sum of $\frac{1}{13}$ and $\frac{51}{60}$?

SECTION IV.—DIVISION OF VULGAR FRACTIONS.

221. The word Division throughout Integral Arithmetic conveys the notion of diminution; for the answer to a Division sum is always as much less than the dividend as the divisor is greater than one. As the divisor is diminished the quotient is increased; therefore if the divisor be less than unity the dividend will be less than the quotient: and this is the case whenever a proper fraction is the divisor. Hence dividing by a fraction increases the dividend just as

multiplying by a fraction diminishes the multiplicand. The definitions of Division given in (88) need only be very slightly extended to meet this case.

222. To divide one number by another is—

I. To find how many times *or parts of a time* the divisor is contained in the dividend.

II. To find a number which is as many times more *or less* than the dividend as unity is more *or less* than the divisor.

III. To find a multiplier which if applied to the divisor would produce the dividend.

223. Before proceeding to the examination of the Rule it is necessary to understand the meaning of the term RECIPROCAL. 1 multiplied by 20 is the reciprocal of 1 divided by 20, or $\frac{1}{20}$, and *vice versâ.* So if 1 be multiplied by 7 and divided by 9 the fraction $\frac{7}{9}$ is the result; but if instead of this 1 were multiplied by 9 and divided by 7 the fraction $\frac{9}{7}$ would result. Now $\frac{7}{9}$ and $\frac{9}{7}$ are each the reciprocal of the other, and generally the fraction inverted is the reciprocal of the other. Thus $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$; 16 of $\frac{1}{16}$; $\frac{11}{100}$ of $\frac{100}{11}$.

224. *If the relation of one magnitude to another is expressed by a number, the relation of the second to the first is expressed by the reciprocal of that number.*

Demonstrative Example.—Because a florin is $\frac{4}{5}$ of half-a-crown, half-a-crown $= \frac{5}{4}$ of a florin; and because $12 = \frac{3}{4}$ of 16, $16 = \frac{4}{3}$ of 12.

General Formula.—If x be $\frac{m}{n}$ of y, then y is $\frac{n}{m}$ of x.

EXERCISE LXIV.

What are the reciprocals of the following numbers:—

1. 7; $\frac{1}{9}$; $\frac{14}{15}$. 2. $2\frac{3}{5}$; $\frac{17}{8}$; $15\frac{4}{11}$. 3. $21\frac{1}{9}$; $\frac{5}{7}$; $108\frac{1}{8}$.

225. *We divide by a fraction when we multiply by its reciprocal.*

Demonstrative Example I.—Divide 8 by $\frac{1}{7}$. Now here we have to ascertain how many times $\frac{1}{7}$ is contained in 8. But because $\frac{7}{7}$ make one unit, $\frac{1}{7}$ is contained 7 times in 1; and is therefore contained

7×8 or 56 times in 8. The answer to the sum is therefore 56. For $\frac{1}{7}$ is contained 56 times in 8. And dividing 8 by $\frac{1}{7}$ is the same as multiplying 8 by 7, which is the reciprocal of $\frac{1}{7}$.

226. *Demonstrative Example II.*—Divide 12 by $\frac{3}{5}$, or find how many times $\frac{3}{5}$ are contained in 12. Now because $\frac{1}{5}$ is contained 5 times in 1, it is contained 5×12 or 60 times in 12 times 1; but if $\frac{1}{5}$ is contained 60 times in a given number three fifths must be contained one-third of 60 times in that number. Hence $\frac{12 \times 5}{3} = 20 =$ the number of times $\frac{3}{5}$ are contained in 12. But this is the same answer as would have been obtained by multiplying 12 by $\frac{5}{3}$, which is the reciprocal of $\frac{3}{5}$.

227. *Demonstrative Example III.*—Let it be required to divide $\frac{3}{4}$ by $\frac{12}{15}$; that is, to divide $\frac{3}{4}$ by the fifteenth part of 12. Let us first divide it by 12. Now (192) to divide by 12 is to multiply the denominator by 12. Hence $\frac{3}{4} \div 12 = \frac{3}{4 \times 12} = \frac{3}{48}$. But it was not required to divide by 12, but by the fifteenth part of 12; therefore in dividing by 12 we have made it 15 times too little; $\frac{3}{48}$ must therefore be multiplied by 15 to rectify this error. But to multiply a fraction by a whole number is to multiply the numerator. Hence $\frac{3}{48} \times 15 = \frac{3 \times 15}{48} = \frac{45}{48}$. But this fraction equals $\frac{3}{4}$ multiplied by $\frac{15}{12}$ or by the reciprocal of $\frac{12}{15}$.

228. *Demonstrative Example IV.*—To divide $\frac{2}{13}$ by $\frac{3}{20}$ is (222) to find how often the latter is contained in the former. To do this we may compare their magnitudes. Let them be brought (203) to a common denominator, then the sum stands—Divide $\frac{40}{260}$ by $\frac{39}{260}$. But to find how many times $\frac{40}{260}$ contains $\frac{39}{260}$ is the same thing as to find how many times 40 oranges contains 39 oranges, or how many times £40 contain £39, or how many times 40 contains 39. Now (182) the fraction $\frac{40}{39}$ represents the number of times that 40 contains 39. Wherefore $\frac{2}{13} \div \frac{3}{20} = \frac{40}{39}$. But this is the same result as $\frac{2}{13} \times \frac{20}{3}$.

229. *Demonstrative Example V.*—It is required to divide $\frac{5}{8}$ by $\frac{3}{11}$. Now (222) this is to find a number which, if multiplied by $\frac{3}{11}$, will give $\frac{5}{8}$ as product. Therefore (213) $\frac{3}{11}$ of this unknown fraction

must equal $\frac{8}{9}$. But by (224) if $\frac{8}{9} = \frac{3}{11}$ of a certain required fraction, that required fraction must equal $\frac{11}{3}$ of $\frac{8}{9}$. But $\frac{11}{3}$ of $\frac{8}{9} = \frac{11}{3} \times \frac{8}{9}$, or $\frac{88}{27}$. Therefore $\frac{8}{9} \div \frac{3}{11} = \frac{8}{9} \times \frac{11}{3}$.

General Formula.—$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$.

RULE TO DIVIDE BY A FRACTION.

230. Find the reciprocal of the divisor (Ex. LXIV.), and then multiply the dividend by it.

Observation.—Sometimes a question in Division takes this form: Resolve $\frac{\frac{27}{5}}{\frac{8}{9}}$ into a simple fraction. Fractions in this shape are often called Complex Fractions. But as it is evidently intended that the upper fraction should be divided by the lower, the sum $\frac{\frac{27}{5}}{\frac{8}{9}}$ is the same as $\frac{27}{5} \div \frac{8}{9}$, or $\frac{27}{5} \times \frac{9}{8}$; and no new rule or explanation is needed in this case. Such a complex fraction would be read $\frac{27}{5}$ *upon* or *by* $\frac{8}{9}$.

EXERCISE LXV.

Solve the following expressions:—

1. $\frac{2}{7} \div \frac{5}{9}$; $\frac{4}{10} \div \frac{3}{13}$; $\frac{12}{27} \div \frac{6}{19}$.
2. $2\frac{1}{7} \div 1\frac{3}{5}$; $41\frac{1}{10} \div \frac{15}{16}$; $\frac{27}{5} \div \frac{14}{15}$.
3. $\frac{\frac{2}{7}}{\frac{4}{9}}$; $\frac{\frac{3}{8} \text{ of } \frac{5}{6}}{\frac{1}{8} + \frac{1}{9}}$; $\frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{2} - \frac{1}{3}}$; $(3\frac{3}{4} + 18\frac{1}{5}) \div (\frac{2}{7} \text{ of } \frac{19}{21})$.
4. $\frac{14\frac{1}{8} + \frac{1}{6}}{2\frac{1}{3} + \frac{5}{9}}$; $\frac{2\frac{1}{4} \times 3\frac{1}{7}}{\frac{3}{7} \text{ of } \frac{2}{9}}$; $\frac{26\frac{3}{7} - 1\frac{13}{21}}{\frac{4}{9} + \frac{6}{5} - \frac{1}{3}}$.
5. $28\frac{1}{3} \div 16\frac{1}{3}$; $[108\frac{3}{12} + (\frac{4}{7} \text{ of } \frac{3}{8}) - 2\frac{1}{9}] - \frac{6}{13} \times \frac{3}{4}$.
6. Divide the product of $\frac{2}{3}$ and $\frac{4}{13}$ by their sum.
7. What number multiplied by $\frac{7}{8}$ will give $2\frac{3}{7}$?
8. What fraction divided by $3\frac{5}{13}$ will give $7\frac{1}{4}$?
9. Divide the product of $\frac{7}{13}$ and $\frac{2}{3}$ by the reciprocal of their sum.
10. To the sum of $\frac{1}{8}$, $\frac{1}{9}$, and $\frac{1}{10}$, add the reciprocal of $15\frac{1}{19}$.
11. From 18 take its fourth, its seventh, and its eleventh, and divide the result by the product of $4\frac{7}{8}$ and $2\frac{1}{9}$.
12. *To* $\frac{7}{12}$ *of* 30 add $\frac{5}{16}$ of $80\frac{1}{2}$, and divide the result by $4\frac{7}{9}$.

ECTION V.—REDUCTION OF VULGAR FRACTIONS TO OTHERS OF DIFFERENT DENOMINATIONS.

231. It is often useful to be able to represent the same quantity fractionally in a variety of different ways—thus: to find what fraction of a pound $\frac{5}{7}$ of a shilling is; or to find what part of a shilling is equal to $\frac{2}{7}$ of a pound. The rule for solving such questions depends on the following considerations.

232. *If one quantity be a certain fractional part of another, it is a greater fractional part of that which is less than that other, and a smaller fractional part of whatever is greater than that other.*

Demonstrative Example.—If a certain sum of money be $\frac{2}{7}$ of £1, it is more than $\frac{2}{7}$ of that which is less than £1, and it is less than $\frac{2}{7}$ of that which is greater than £1. Or $\frac{2}{7}$ of £1 is more than $\frac{2}{7}$ of a shilling, but it is less than $\frac{2}{7}$ of £5.

It is evident also that a sum of money which is a certain fraction of a pound is a greater fraction of a shilling in just the same degree as the shilling is less than a pound. That is to say, $\frac{2}{7}$ of £1 is $\frac{2}{7}$ of 20s., and is therefore 20 times $\frac{2}{7}$ of 1s., or $\frac{20 \times 2}{7}$, $\frac{40}{7}$ of 1s. So also, if a certain sum of money is $\frac{4}{5}$ of a shilling, it is a less fraction of £1 in just the same degree as a shilling is less than a pound; that is, because $\frac{4}{5}$ of a shilling is $\frac{4}{5}$ of $\frac{1}{20}$ of a pound, $\frac{4}{5}$ of a shilling $= \frac{4}{5 \times 20}$ or $\frac{4}{100}$ of £1.

General Formula.—If a be $\frac{n}{m}$ of b it is $\frac{pn}{m}$ of $\frac{b}{p}$, and it is $\frac{n}{pm}$ of pb.

RULE TO EXPRESS FRACTIONS OF A CERTAIN NUMBER BY EQUIVALENT FRACTIONS OF ANOTHER NUMBER OF THE SAME KIND.

233. When the fraction is to be altered to one of a higher name, multiply the denominator by as many of the less as make one of the greater.

But when the fraction is to be altered to an equivalent fraction of a lower name, multiply the numerator

by as many of the lower name as make one of the higher.

Example I.—What fraction of a yard is $\frac{3}{5}$ of an inch?

Here because $\frac{3}{5}$ of an inch has to be expressed as a fraction of a length greater than an inch, the new fraction must be as many times less than $\frac{3}{5}$ as an inch is less than a yard; that is to say, $\frac{3}{5}$ of an inch $= \frac{3}{5}$ of $\frac{1}{36} = \frac{3}{5 \times 36} = \frac{3}{180}$ of a yard.

Example II.—What fraction of an hour is $\frac{2}{11}$ of a day?

Here because a day is 24 times greater than an hour any fraction of a day is 24 times that fraction of an hour. Hence $\frac{2}{11}$ of a day $=$ $\frac{2}{11}$ of 24 hours $= \frac{2 \times 24}{11}$, or $\frac{48}{11}$ of an hour.

Exercise LXVI.

1. Reduce $\frac{3}{7}$ of a day to the fraction of a year, and of a lunar month.
2. Reduce $\frac{5}{9}$ of a furlong to the fraction of a league, a mile, a pole, a yard, and a foot.
3. How would $\frac{3}{7}$ of a florin be expressed as a fraction of each of the English coins?
4. Reduce $\frac{5}{18}$ of an acre to the fraction of a square mile, and also of an acre.
5. Reduce $\frac{14}{17}$ of an avoirdupois ounce to the fraction of a cwt., a pound, a troy pound, a troy ounce, and a grain.
6. What portion of 7 yards is $\frac{5}{11}$ of an inch?
7. What fraction of 9 inches is $\frac{5}{73}$ of a mile?
8. If a certain sum of money be $\frac{41}{55}$ of 10s. what fraction is it of 15s., of £1, and of a five pound note?
9. Express $\frac{8}{13}$ of an hour in terms of a minute, a day, and a week.
10. Express $\frac{18}{25}$ of a gallon in terms of a pint, a puncheon, and a pipe.
11. What part of 100 quarters of wheat is $\frac{5}{7}$ of 9 bushels?
12. What part of 17 square miles is $\frac{3}{8}$ of $\frac{5}{6}$ of 12 acres?
13. Express 2 roods 15 perches as the fraction of an acre, and of a square mile.
14. What part of £50 is $\frac{3}{8}$ of $\frac{3}{10}$ of 2 shillings?

SECTION VI.—CONTINUED FRACTIONS.

234. When a vulgar fraction is expressed by two large numbers which are prime to each other, it cannot (197) be reduced to a lower name, and yet it is not easy to acquire an exact idea of its value. The process of resolving such an expression into the form of a continued or converging fraction is designed to give a series of fractions expressed in smaller numbers, but approximating as nearly as possible to the value of the given fraction.

The terms of the fraction $\frac{135}{292}$ for example are irreducible. We may by (194) divide both by 135. Hence* $\frac{135}{292} = \frac{1}{(\frac{292}{135})}$. Here 1 is the numerator, and the denominator is an improper fraction, which by (185) may itself be reduced to a mixed number. Thus we have $\frac{135}{292} = \frac{1}{2\frac{22}{135}}$. Suppose now we neglect the fraction $\frac{22}{135}$, the remaining fraction $\frac{1}{2}$ is a rough approximation to $\frac{135}{292}$, but it is rather too great, because the whole of the denominator has not been taken into account. But if for $\frac{22}{135}$ we substitute 1, the fraction becomes $\frac{1}{2+1}$ or $\frac{1}{3}$; but this is evidently less than the required fraction, which therefore lies between $\frac{1}{2}$ and $\frac{1}{3}$.

In order to obtain a closer approximation we treat the last fraction in the same manner as the former one. Thus $\frac{22}{135} = \frac{1}{(\frac{135}{22})} = \frac{1}{6 + \frac{3}{22}}$ and the proposed fraction $\dfrac{135}{292} = \cfrac{1}{2 + \cfrac{1}{6 + \frac{3}{22}}}$. If we now neglect the $\frac{3}{22}$ we observe that the remaining fraction $\frac{1}{2 + \frac{1}{6}}$ or $\frac{1}{(\frac{13}{6})}$ or $\frac{6}{13}$ is too little, for the denominator is too great. Hence we infer that the value of the original fraction lies between $\frac{1}{2}$ and $\frac{6}{13}$. But as (210)

* This transformation may be justified also on the general principle explained in (223) concerning the reciprocals of numbers. *Every number whether integral or fractional equals unity divided by its reciprocal, e.g.*, $6 = \frac{1}{\frac{1}{6}}$; $\frac{4}{7} = \frac{1}{\frac{7}{4}}$, &c.

the difference between them is $\frac{1}{26}$, the fraction $\frac{6}{13}$ does not differ from the truth by so much as $\frac{1}{26}$.

If we now deal with the fraction $\frac{3}{22}$ as with the rest, we have this result $\frac{135}{292} = \cfrac{1}{2+\cfrac{1}{6+\cfrac{1}{(\frac{22}{3})}}} = \cfrac{1}{2+\cfrac{1}{6+\cfrac{1}{7+\cfrac{1}{3}}}}$. If now we neglect the $\frac{1}{3}$ and consider the rest of the expression only, we have $6\frac{1}{7} = \frac{43}{7}$ therefore $\frac{1}{6\frac{1}{7}} = \frac{1}{(\frac{43}{7})}$ or $\frac{7}{43}$ and $2 + \frac{1}{6\frac{1}{7}} = 2 + \frac{7}{43}$ or $\frac{93}{43}$. Hence $\cfrac{1}{2+\cfrac{1}{6+\cfrac{1}{7}}} = \frac{1}{(\frac{93}{43})} = \frac{43}{93}$. Now the last expression $\frac{6}{13}$ was less than $\frac{135}{292}$ because it made the denominator appear greater than it really was. But since $\frac{1}{7}$ is greater than $\frac{1}{7+\frac{1}{3}}$, therefore $\cfrac{1}{6+\cfrac{1}{7}}$ is less than $\cfrac{1}{6+\cfrac{1}{7+\cfrac{1}{3}}}$, and is consequently greater than $\cfrac{1}{2+\cfrac{1}{6+\cfrac{1}{7+\cfrac{1}{3}}}}$. The new fraction $\frac{43}{93}$ is, therefore, too great, and the real value of $\frac{135}{292}$ lies between $\frac{6}{13}$ and $\frac{43}{93}$, but as these differ only by $\frac{1}{1209}$ the error committed in calling the original fraction $\frac{43}{93}$ is less than $\frac{1}{1209}$.

The fraction $\frac{135}{292}$ has therefore been resolved into this form, and

$$\cfrac{1}{2+\cfrac{1}{6+\cfrac{1}{7+\cfrac{1}{3}}}}$$

if we begin at the upper fraction, and take the several denominators successively into account, we have the series of fractions, $\frac{1}{2}, \frac{6}{13}, \frac{43}{93}, \frac{135}{292}$, of which the second is nearer the truth than the first, and the third nearer than the second; but each is nearer than any other expression formed of numbers equally small.

A continued fraction therefore always has unity for its numerator, and for a denominator a whole number plus a fraction, which has itself unity for a numerator, and for a denominator a whole number plus another similar fraction, and so on.

235. It is evident that the series of denominators in the case given might have been at once obtained by employing the method (155) for finding the greatest common measure of two numbers. Here the series of quotients, 2, 6, 7, and 3, are the denominators required, and the four converging fractions are thus formed.

$$\begin{array}{l}
135)\,292\,(2\\
\quad\ \ \underline{270}\\
\quad\quad 22)\,135\,(6\\
\quad\quad\quad\ \ \underline{132}\\
\quad\quad\quad\quad 3)\,22\,(7\\
\quad\quad\quad\quad\quad \underline{21}\\
\quad\quad\quad\quad\quad 1)\,3\,(3
\end{array}$$

$$\text{First} = \frac{1}{\text{First quotient}} = \frac{1}{2}$$

$$\text{Second} = \frac{1 \times 6}{(2 \times 6) + 1} = \frac{6}{13}$$

$$\text{Third} = \frac{(6 \times 7) + 1}{(13 \times 7) + 2} = \frac{43}{93}$$

$$\text{Fourth} = \frac{(43 \times 3) + 6}{(93 \times 3) + 13} = \frac{135}{292}$$

The first fraction has its numerator and denominator multiplied by the second quotient, and one added to the denominator in order to make the second fraction. This second fraction has its numerator and denominator multiplied by the third quotient: to these results the numerator and denominator of the first fraction are added, and thus the third fraction is found. The other fractions are obtained by the same method.

To resolve a fraction into a convergent series—

Rule.

236. Transform the fraction into 1 divided by its reciprocal. Resolve this reciprocal into a mixed number, and transform the fractional part of this number into unity divided by its reciprocal. Proceed in this way until 1 is the numerator of the last fraction.

Or, with the numerator and denominator of the fraction proceed as in the rule for finding the greatest common measure; the series of quotients will be the series of denominators required.

Example.—Reduce $\frac{363}{149}$ to the form of a continued fraction.

$$\frac{363}{149} = 2\tfrac{65}{149} = 2 + \cfrac{1}{\left(\frac{149}{65}\right)} = 2 + \cfrac{1}{2\left(\frac{19}{65}\right)} \qquad \text{1st approx. } \frac{5}{2}$$

$$= 2 + \cfrac{1}{2 + \cfrac{1}{\left(\frac{65}{19}\right)}} = 2 + \cfrac{1}{2 + \cfrac{1}{3 + \left(\frac{8}{19}\right)}} \qquad \text{2nd approx. } \frac{17}{7}$$

$$= 2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{\left(\frac{19}{8}\right)}}} = 2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \left(\frac{3}{8}\right)}}} \qquad \text{3rd approx. } \frac{39}{16}$$

$$= 2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{\left(\frac{8}{3}\right)}}}} = 2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{2 + \left(\frac{2}{3}\right)}}}} \qquad \text{4th approx. } \frac{95}{39}$$

$$= 2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\left(\frac{3}{2}\right)}}}}} = 2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2}}}}}} \qquad \text{5th approx. } \frac{134}{55}$$

Observation.—In each case the approximating fraction is to be obtained by calculating the value of the whole expression, omitting the lowest fraction at that stage of the process.

EXERCISE LXVII.

Convert each of the following fractions into a continuous form, and find the series of convergents.

1.	$\frac{829}{347}$;	$\frac{159}{493}$.	2.	$\frac{1104}{887}$;	$\frac{173}{200}$.
3.	$\frac{425}{1717}$;	$\frac{603}{138}$.	4.	$\frac{113}{764}$;	$\frac{456}{121}$.
5.	$\frac{2602}{58363}$;	$\frac{503}{100103}$.	6.	$\frac{211}{417}$;	$\frac{718}{125}$.
7.	$\frac{1845}{1717}$;	$\frac{427}{365}$.	8.	$\frac{68}{99}$;	$\frac{427}{365}$.

SECTION VII.—MISCELLANEOUS APPLICATIONS OF VULGAR FRACTIONS.

237. In many parts of arithmetic, fractional expressions occur, which require the foregoing rules to solve them. Such cases assume very varied forms, and are generally comprehended under one or other of the following cases.

238. CASE I.—WHEN THE VALUE OF THE WHOLE QUANTITY IS GIVEN, TO FIND THE NEAREST INTEGRAL VALUE FOR A FRACTION—

RULE.

Multiply the quantity by the numerator and divide by the denominator.

Example.—What is the value of $\frac{5}{8}$ of £1. Here $\frac{1}{8}$ of £1 = 2s. 6d. Therefore $\frac{5}{8}$ of £1 must equal five times 2s. 6d., or 12s. 6d. Or, because $\frac{5}{8}$ of £1 = $\frac{1}{8}$ of £5, therefore £5 ÷ 8, or 12s. 6d., is the answer.

In either case it will be seen that the £1 is multiplied by 5 and divided by 8, and that whether we divide first and then multiply, or multiply first and divide afterwards, the answer is the same.

239. *Observation.*—Suppose it is required to find $\frac{7}{11}$ of 200. If we here try to take $\frac{1}{11}$ of 200, and then multiply by 7, we first obtain $200 \div 11 = 18\frac{2}{11}$, and this multiplied by 7 gives $126\frac{14}{11}$, or $127\frac{3}{11}$. We have here had to deal with fractions in both lines, and it would have been simpler to multiply the 200 by 7 in the first instance, and then divide this product by 11; for $\frac{7}{11}$ of 200 is the same as $\frac{1}{11}$ of 7×200, or $\frac{1400}{11} = 127\frac{3}{11}$.

So if it be required to find $\frac{8}{9}$ of 2s. 6d., this may either be found by taking $\frac{1}{9}$ of 2s. 6d. and multiplying it by 8, or by taking 8 times 2s. 6d. and dividing it by 9.

	s.	*d.*
9)	2	6
		$3\frac{1}{4}$ + $\frac{1}{3}$ of a farthing
		8
	2	$2\frac{1}{2}$ + $\frac{2}{3}$ of a farthing

	s.	*d.*
	2	6
		8
9)	20	0
	2	$2\frac{1}{2}$ + $\frac{2}{3}$ of a farthing.

Now it is evident that the second of these two methods is preferable, as by it the answer is obtained with less trouble. Hence it should be remembered—

In all cases in which it is required to multiply by one number and divide by another, it is more convenient to perform the multiplication first and the division afterwards.

Exercise LXVIII.

1. Find $\frac{7}{8}$ of 2s. $7\frac{1}{2}$d.; $\frac{14}{15}$ of $\frac{3}{4}$ of £5.
2. $\frac{5}{7}$ of 1 cwt.; and $\frac{3}{5}$ of $\frac{1}{9}$ of a ton.
3. $\frac{2}{3}$ of $\frac{7}{9}$ of the product of 12 and 9; $\frac{14}{25}$ of $5\frac{1}{2}$ miles.
4. $\frac{7}{16}$ of $\frac{12}{19}$ of £50; $\frac{16}{23}$ of £7 3s. 4d.
5. $\frac{7}{8}$ of £4 3s. 9d.; $\frac{7}{12}$ of £18 6s. 8d.
6. $\frac{4}{5}$ of $\frac{20}{19}$ of £81 5s. 4d.; $\frac{2}{7}$ of $\frac{3}{5}$ of £75 10s.
7. $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{7}{8}$ of 20 guineas; $\frac{5}{7}$ of $\frac{3}{14}$ of $\frac{8}{9}$ of £205.
8. $5\frac{4}{9}$ of a hhd. of ale; $17\frac{2}{3}$ of a hhd. of wine.
9. $\frac{4}{7}$ of 2 miles; $\frac{13}{15}$ of an acre; $\frac{7}{12}$ of a square mile.
10. $5\frac{7}{8}$ of a bushel; $\frac{4}{17}$ of a quarter; $12\frac{5}{7}$ of a gallon.
11. $\frac{4}{7}$ of $\frac{2}{19}$ of £20; $\frac{8}{33}$ of £11; $\frac{2}{7}$ of $\frac{5}{9}$ of a florin.
12. $\frac{5}{7}$ of a lunar month; $\frac{2}{13}$ of a year.
13. $\frac{2}{7}$ of $\frac{3}{5}$ of 4 miles; $\frac{4}{11}$ of $\frac{14}{17}$ of a league.
14. $\frac{3}{5}$ of $\frac{4}{17}$ of 2 tons; $\frac{7}{10}$ of $\frac{2}{11}$ of 4 cwt.
15. *Add together* $\frac{7}{11}$ of £1, $\frac{2}{13}$ of a crown, and $\frac{4}{17}$ of a guinea.

240. CASE II.—WHEN THE VALUE OF A FRACTION IS GIVEN, TO FIND THE WHOLE QUANTITY OF WHICH IT IS THE FRACTION—

RULE.

Multiply the given value by the denominator and divide by the numerator.

Example I.—What is the number of which 30 is $\frac{5}{7}$? Here if 30 is $\frac{5}{7}$ of the required number, the fifth part of 30 must be $\frac{1}{7}$ of that number. But the fifth of 30 $= 6$; wherefore 6 is $\frac{1}{7}$ of the answer. But 6 is $\frac{1}{7}$ of 6×7 or 42; and 42 is the answer, for 30 is $\frac{5}{7}$ of 42.

Example II.—What is the number of which x is $\frac{a}{b}$? Here it is obvious that the whole number required must be as much greater or lesss than x as b is greater or less than a. Wherefore $\frac{x\,b}{a} =$ the answer required.

Example III.—What is the sum of money of which 5s. is $\frac{10}{17}$? Here if 5s. is 10 seventeenths of the required sum, a tenth of 5s. must be one seventeenth. But 5s. $\div$ 10 = 6d.; therefore 6d. = $\frac{1}{17}$ of the required sum. But 6d. $\times$ 17 = 102d. = 8s. 6d.; wherefore 8s. 6d. is the sum of which 5s. is $\frac{10}{17}$.

EXERCISE LXIX.

1. What is the period of which 3 hours 20 mins. is $\frac{5}{9}$? of which 3 days is $\frac{6}{7}$?
2. What sum is that of which 3s. 6d. is $\frac{14}{17}$? of which £7 12s. 6d. is $\frac{5}{9}$?
3. What length is that of which 25 yards is $\frac{15}{27}$? of which 7 feet is $\frac{21}{100}$?
4. Of what weight is 2 oz. 3 dwts. three seventeenths?
5. What weight is that of which 17 lbs. 2 oz. is $\frac{12}{13}$? of which $5\frac{1}{2}$ cwt. is $\frac{9}{11}$?
6. Of what number is 17 equal to $\frac{15}{23}$? of what is $5\frac{1}{5}$ equal to $\frac{22}{34}$?
7. What is the whole of which £5 12s. 6d. is $\frac{3}{8}$?
8. 17 men have equal allotments of land in a field, the space of 5 of them amounts to 7 roods 20 perches; what is the area of the field?

241. CASE III.—TO EXPRESS A GIVEN QUANTITY AS A FRACTION OF ANOTHER OF THE SAME KIND—

RULE.

Reduce both to the same name if concrete numbers; the number contained in the given quantity, and that in the other of the same kind, will be the numerator and denominator of the fraction required.

Example I.—What fraction is 20 of 29? According to the Definition of Fractions (182) 20 is $\frac{20}{29}$ of 29. Questions of this kind, in which abstract numbers only are concerned, can readily be solved by making one of the numbers a numerator and the other a denominator, for whether a be greater or less than b, it is always true that $\frac{a}{b}$ represents the fraction a is of b.

But when the numbers concerned are concrete this rule cannot be adopted unless they refer to the same name. Thus the fraction 13 feet is of 20 yards is not $\frac{13}{20}$. But after reducing the yards to 60 feet, the question becomes "What fraction is 13 feet of 60 feet?" and the answer is clearly $\frac{13}{60}$.

Example II.—What fraction of 15s. $9\frac{1}{2}$d. is $2\frac{3}{4}$d.?

By Rule of Descending Reduction (87) 15s. $9\frac{1}{2}$d. $=$ 758 farthings, and $2\frac{3}{4}$d. $=$ 11 farthings. Hence $2\frac{3}{4}$d. $= \frac{11}{758}$ of 15s. $9\frac{1}{2}$d.

EXERCISE LXX.

1. What fraction of $4\frac{1}{2}$ yards is $3\frac{1}{4}$ inches?
2. What fraction of a guinea is 3s. $4\frac{1}{4}$d.?
3. What fraction of a cwt. is 2 lbs. $6\frac{1}{2}$ oz.?
4. What fraction of 5 tons is 3 cwt. 2 qrs. 17 lbs.?
5. What fraction of 5 weeks is 2 hours 20 mins.?
6. Express £17 3s. 10d. and £2 3s. 8d. as fractions of £100.
7. Express 2 gallons 1 pint as a fraction of a barrel of beer.
8. Express 3s. $4\frac{1}{2}$d. as fractions of a crown and a guinea.
9. If a mountain be $4\frac{7}{8}$ miles high, express its altitude as a fraction of the earth's diameter, which is 7926 miles.
10. A parish contains 7233 acres, 29 poles, express its area as a fraction of the whole of England, which contains 58000 sq. miles.

PRACTICE.

242. In this Rule it is required to find by the help of fractions the value of any number of articles when the price of one is known.

Observation.—In (138) it was shown that if a number increased any number of times made up another, the first was called a *measure* of the second, and the second a *multiple* of the first. But these terms are seldom used except in the case of abstract numbers. When any *concrete* number taken a certain number of times makes up another, the first is commonly called an *Aliquot part* of the second. Thus 5s. is an aliquot part of £1; 14 lbs. is an aliquot part of a hundredweight; 2 ounces of a pound, &c., &c. But 3s. is not an aliquot part of £1; nor is 7 ounces an aliquot part of a pound.

Example I.—Suppose it is required to find the price of 2834 articles at 17s. 10½d. each.

Now 2834 articles at £1 each would be worth £2834.

	s.	*d.*			£	*s.*	*d.*		£	*s.*	*d.*
Cost of 2834 articles at	10	0	each	=	1417	0	0	or ½ of	2834	0	0
Cost of 2834 articles at	5	0	each	=	708	10	0	or ½ of	1417	0	0
Cost of 2834 articles at	2	6	each	=	354	5	0	or ½ of	708	10	0
Cost of 2834 articles at		3	each	=	35	8	6	or $\frac{1}{10}$ of	354	5	0
Cost of 2834 articles at		1½	each	=	17	14	3	or ½ of	35	8	6
Cost of 2834 articles at	17	10½	each	=	2532	17	9	or the sum of the several answers.			

The same sum might have been worked thus:—

d.		*s.*	*d.*		
6	2	2834		=	cost of 2834 articles at 1s. each
		17			
		48178		=	cost at 17s.
3	2	1417		=	cost at 6d. or half the cost at 1s.
1½	2	708	6	=	cost at 3d. or half the cost at 6d.
		354	3	=	cost at 1½d. or half the cost at 3d.
		20) 50657	9	=	cost at 17s. 10½d.
		£2532 17	9		Answer reduced to pounds.

Or if 2834 pence were taken as the standard, the number might have been multiplied by the number of pence in 17s. 10d. Then half the upper line might have been taken to give the value of the articles at ½ each, and with this added, the answer would appear in pence.

243. No rule can be given for determining what "aliquot" parts should be chosen. A little practice will soon enable a learner to select the most convenient. Three things only require to be remembered.

I. The number of articles given may be taken to represent the number of pounds, shillings, pence, or farthings, which so many articles would cost at a pound or shilling, a penny or a farthing each.

II. This number must be multiplied or divided according as the answer required is to be greater or less.

III. Each line will represent the cost of the given number of articles at a certain part of the price each, and the sum of all the lines, or the answer, will represent the cost of the given number of articles at the whole of the price each.

RULE FOR PRACTICE.

244. Multiply the number of articles by the number of pounds or shillings in the price, and take aliquot parts for the rest.

Example II.—Find the value of 8632 articles at £1 14s. 3¾d.

			£	*s.*	*d.*		
10s.	= £1	÷ 2	8632	0	0	=	cost at £1 each
2s. 6d.	= 10s.	÷ 4	4316	0	0	=	cost at 10s. each
1s. 3d.	= 2s. 6d.	÷ 2	1079	0	0	=	cost at 2s. 6d. each
			539	10	0	=	cost at 1s. 3d. each
6d.	= 2s. 6d.	÷ 5	215	16	0	=	cost at 6d. each *
¾d.	= 6d.	÷ 8	26	19	6	=	cost at ¾d. each
			14809	5	6	=	cost at £1 14s. 3¾d.

Example III.—Find the value of 23571 articles at 6s. 7$\frac{5}{12}$d. each.

		s.	*d.*		
6	½	23571			
		6			
		141426		=	price at 6s.
1d.	$\frac{1}{12}$	11785	6	=	price at 6d.
		982	1¼	=	price at 1d.
$\frac{5}{12}$ of 1d	$\frac{5}{12}$	409	2$\frac{15}{24}$	=	price at $\frac{5}{12}$ of 1d.
		20) 154602	10$\frac{3}{24}$	=	price at 6s. 7$\frac{5}{12}$d.
		£7730 2	10$\frac{3}{24}$		

* Observe that this line has not been obtained from that immediately above it, but *from the preceding,* because 6d is an aliquot part of 2s. 6d., but not of 1s. 3d.

Exercise LXXI.

1. 7246 articles at £7 8s. 9d.; 2397 at £2 6s. 8¾d.
2. 4096 at £17 3s. 5d.; 50832 at 4s. 3¼d.
3. 20738 at £6 6s. 0¾d.; 10096 at 17s. 2¼d.
4. 1309 at £25 4s. 7d.; 70862 at £1 12s. 6¾d.
5. 20891 at £14 2s. 11¼d.; 35619 at £7 13s. 3½d.
6. 1057 at 10s. 6d.; 20731 at 15s. 6d.
7. 16379 at 5s. 4¼d.; 41986 at £1 10s. 4¼d.
8. 12357 at 11¾d.; 32705 at 6s. 7½d.
9. 1178 at £5 6s. 3¼d.; 20196 at £15 3s. 5d.
10. 36917 at 7s. 2¼d.; 51398 at 15s. 10d.
11. 37128 at 14s. 6½d.; 18603 at 17s. 11½d.
12. 41357 at £53 7s. 3¼d; 2198 at £16 2s. 7d.
13. 218763 at 10¼d.; 41378 at 7¾d.
14. 71639 at £4 17s. 2d.; 21086 at £3 12s. 3d.
15. 10397 at £5 6s. 3d.; 2178 at 17s. 10½d.
16. 3147 at 1s. 9½d.; 41263 at 14s. 6¼d.
17. 21076 at £15 11s. 2½d.; 1413 at £12 10s. 7¼d.
18. 17621 at £1 16s. 3¾d.; 2108 at £1 6s. 2¼d.
19. 30128 at £19 4s. 2d.; 17186 at £12 13s. 8d.
20. 10719 at £43 2s. 5d.; 20165 at £57 19s. 4¼d.

245. When the number of times the price has to be taken is not simply expressed, but is shown by numbers of different denominations, the question is said to be in Compound Practice.

Example I.—What is the value of 2 cwt. 3 qrs. 16 lbs. at £14 7s. 6d. per cwt.?

	£	*s.*	*d.*	
2 qrs. = 1 cwt. ÷ 2	14	7	6	= value of 1 cwt.
			2	
	28	15	0	= value of 2 cwt.
1 qr. = 2 qrs. ÷ 2	7	3	9	= value of 2 qrs.
14 lbs. = 1 qr. ÷ 2	3	11	10½	= value of 1 qr.
2 lbs. = 14 lbs. ÷ 7	1	15	11¼	= value of 14 lbs.
		5	$1\frac{19}{28}$	= value of 2 lbs.
	41	11	$8\frac{26}{28}$	= value of 2 cwt. 3 qrs. 16 lbs.

Example II.—How much must be paid for 27 acres 3 roods 27 poles at £7 10s. 8d. per acre?

	£	s.	d.	
2 roods = $\frac{1}{2}$	7	10	8	= price of 1 acre
			9	
	67	16	0	= price of 9 acres
			3	
	203	8	0	= price of 27 acres
1 rood = $\frac{1}{2}$	3	15	4	= price of 2 roods
20 poles = $\frac{1}{2}$	1	17	8	= price of 1 rood
5 poles = $\frac{1}{4}$		18	10	= price of 20 poles
2 poles = $\frac{1}{10}$ of 20 poles		4	$8\frac{1}{2}$	= price of 5 poles
		1	$10\frac{6}{10}$	= price of 2 poles
	210	6	$5\frac{1}{10}$	= price of 27 a. 3 r. 27 p.

Exercise LXXII.

1. 2 cwt. 3 qrs. 10 lbs. at £1 16s. 6d. per cwt.
2. 17 cwt. 1 qr. 15 lbs. 7 oz. at £2 15s. 3d. per cwt.
3. 3 tons 14 cwt. 11 lbs. at £7 3s. 8d. per cwt.
4. 3 qrs. 17 lbs. 9 oz. at £10 per cwt.
5. 5 acres 2 roods 27 poles at £5 10s. per acre.
6. 27 acres 1 rood 9 poles at £7 12s. per acre.
7. 13 acres 3 roods 33 poles at £11 per acre.
8. 200 acres 2 roods 17 poles at 11s. 6d. per rood.
9. 18 lbs. 13 oz. at £100 per ton.
10. 7 gallons 3 pints at £1 7s. per barrel.
11. 3 quarts $1\frac{1}{2}$ pints at £2 10s. per butt.
12. 1 bush. 3 pecks at £2 18s. per quarter.
13. Find the wages for 17 months 4 weeks 9 days at £2 10s. 6d. per month.
14. Rent of 246 acres 2 roods 15 poles at £1 7s. 6d. per acre.
15. 15 cwt. 2 qrs. 17 lbs. at £2 6s. 7d. per quarter.
16. 17 cwt. 1 qr. 9 lbs. 6 oz. at 1s. $8\frac{1}{2}$d. per lb.

Questions on Vulgar Fractions.

Distinguish between Integral and Fractional Arithmetic. To which of the former rules may Fractional Arithmetic be considered the sequel? Why? Give the meaning of the words, fraction, integer, numerator, denominator, proper, and improper.

What is a mixed number, and what change may it always undergo? Give an

example, and state the reason. How may a fraction be multiplied or divided by an integer? Which is the preferable method of dividing $\frac{3}{16}$ by 2, and why?

What changes may every fractional expression undergo? State the principle, and give a demonstrative example. What is the effect on a fraction of equal increase or subtraction to its terms, and why? Suppose $\frac{10}{20}$ be converted into $\frac{1}{2}$, what advantage is gained and what principle is illustrated? Suppose $\frac{5}{7}$ is converted into $\frac{15}{21}$, what advantage is gained and what principle illustrated?

In what Rule of Fractions are the rules for finding the greatest common measure and the least common multiple useful, and why? When does the addition of the numerators effect the addition of the fractions? Give the rule for finding a common denominator, and state when and why it is to be used? What truths are assumed in the process of adding $\frac{5}{7}$ to $\frac{5}{8}$? Demonstrate each step of the process.

Define Multiplication in the widest sense of the term. How does Fractional Multiplication differ from Integral? In how many ways can you prove that $\frac{4}{9} \times \frac{3}{5} = \frac{4 \times 3}{9 \times 5}$? Demonstrate the rule by each method. Define Division, and show how Fractional differs from Integral Division. What is a reciprocal? In how many ways can you prove that $\frac{4}{9} \div \frac{3}{5} = \frac{4 \times 5}{9 \times 3}$? Demonstrate the rule by each method.

What is the general rule for reducing a fraction to another of a different denominator? Suppose $a = nb$, what fraction of a is $\frac{7}{8}$ of b, and what fraction of b is $\frac{3}{8}$ of a? Give the reason in each case.

Explain why, in a series of convergent fractions, the several approximations are alternately greater and less than the fraction itself. What general principle is illustrated by the method of resolving fractions into the continued form?

Why is it better to multiply first and divide afterwards, when both have to be done in one sum? In what operation should we divide a quantity by the numerator and multiply by the denominator, and when should the inverse process be used? Give the reason in both cases. What is the general rule to be observed in Practice, and how is it connected with Fractions.

State in words the truth concerning Fractions illustrated by each of the following formulæ:—

1. $\frac{a}{b} \times c = \frac{ac}{b} = \frac{a}{b \div c}$

2. $\frac{a}{b} \div c = \frac{a \div c}{b} = \frac{a}{bc}$

3. $\frac{a}{b} = \frac{ma}{mb} = \frac{a \div m}{b \div m}$

4. $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$

5. $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad + cb}{bd}$

6. $a + \frac{b}{c} = \frac{ac + b}{c}$

7. If x be less than y, $\frac{x}{y}$ is less than $\frac{x + a}{y + a}$ but greater than $\frac{x - a}{y - a}$

If x be greater than y, $\frac{x}{y}$ is greater than $\frac{x + a}{y + a}$ but less than $\frac{x - a}{y - a}$

8. $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$

9. $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{cb}$

10. $\frac{a}{b} = \frac{1}{b \div a}$

11. $\frac{a}{b} \div \frac{c}{b} = a \div c$

General Exercises on Vulgar Fractions.

1. Add together $\frac{5}{9}$ of £1, $\frac{3}{14}$ of a guinea, $\frac{2}{11}$ of a crown, and $\frac{7}{8}$ of a shilling.

2. If two men can do a certain work together in 15 days, and one of them could do it by himself in 25 days, how long would the other be in doing it alone?

3. Find the difference between the square of $4\frac{7}{8}$ and the cube of $2\frac{3}{5}$.

4. What fraction of £7 is equivalent to $\frac{7}{8}$ of a guinea?

5. Express the area of a plot of 3 roods 7 poles, as a fraction of a field of $9\frac{1}{2}$ acres.

6. Of how many pounds of sugar does a tradesman defraud his customers in retailing $4\frac{3}{7}$ cwt. of sugar, if he uses a false weight of $15\frac{5}{8}$ oz. for a pound?

7. What number divided by $\frac{3}{5}$ will give $3\frac{7}{8}$ as quotient? and what number multiplied by $13\frac{4}{7}$ will give $15\frac{9}{13}$ as product?

8. Find the sum of $\frac{2}{3}$, $\frac{1}{7}$, and $\frac{2}{5}$, and the product of $2\frac{1}{3}$ and $3\frac{2}{5}$.

9. Find the difference between the sum of $\frac{3}{5}$ and $\frac{4}{9}$, and the product of $\frac{7}{12}$ and $\frac{8}{11}$.

10. Express the difference between 12 and $11\frac{4}{73}$ as a converging fraction.

11. What is the value of 19 yards 2 feet 5 inches at 5s. 7d. per foot?

12. Find the land tax on 43 acres 2 roods 26 poles at 3s. 7d. per acre.

13. How much of £5 8s. 4d. is $\frac{2}{3}$ of $\frac{1}{8}$ of £20?

14. Divide the product of $4\frac{7}{8}$ and $5\frac{4}{9}$ successively by their sum and their difference.

15. From the sum of $\frac{2}{7}$ and $\frac{4}{17}$ take the difference between $\frac{3}{8}$ and $\frac{5}{11}$.

16. Which is greater, the product of $17\frac{2}{7}$ and $5\frac{3}{8}$, or the product of $\frac{792}{5}$ and $12\frac{3}{8}$, and what is the difference?

17. Find the difference between the continued product of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{5}{6}$, and that of $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, and $\frac{7}{8}$.

18. What is the product of $\frac{2}{3}$ of $\frac{7}{8}$ and $\frac{3}{8}$ of $8\frac{1}{4}$?

19. A boy has a number of marbles, of which he loses $\frac{3}{8}$ at play, gives away $\frac{5}{16}$ to one schoolfellow, and $\frac{1}{4}$ to another, he then has 7 *left; how many* had he at first?

20. A father leaves to one of his children $\frac{1}{5}$ of his property, to another £200, and the remaining $\frac{3}{4}$ among the rest, how much does he leave altogether?

21. If a post be $\frac{1}{3}$ in the water, $\frac{1}{7}$ out of the water, and 22 feet in the mud, what is its length?

22. How much must be added to $3\frac{4}{11}$ to make $7\frac{2}{13}$, and what number taken from $23\frac{5}{19}$ will leave $\frac{5}{7}$ of $9\frac{1}{4}$?

23. Add together $\frac{17}{58}$ of £176 18s., and $\frac{11}{18}$ of £150.

24. What is the value of $\frac{23}{67}$ of a ship, $\frac{5}{8}$ of which is valued at £755 15s.?

25. Divide the difference between $\frac{7}{8}$ and $\frac{5}{14}$ by the sum of $\frac{1}{7}$ and $\frac{5}{8}$.

26. If of a tavern bill of £3 0s. $11\frac{1}{2}$d. my share and that of two others amounts to 16s. $7\frac{1}{2}$d., how many are there in the company?

27. If the area of a table be 76 square inches, and its length $10\frac{5}{8}$ inches, what is its breadth?

28. After spending $\frac{2}{7}$ of the money in my purse I find that $\frac{3}{4}$ of the remainder amounts to 16s. 3d., how much had I at first?

29. Subtract from 5 its third, its fourth, and its fifth, what fraction of $17\frac{1}{2}$ is the remainder?

30. If the product of $\frac{7}{8}$ and $2\frac{1}{4}$ be added to the sum of $21\frac{3}{7}$ and $3\frac{6}{11}$, by how much will the result differ from 100?

31. Compare the magnitudes of $\frac{11}{19}$, $\frac{18}{29}$, and $\frac{23}{38}$.

32. Compare the values of $\frac{1}{19}$ of £1, $\frac{1}{20}$ of a guinea, and $\frac{8}{35}$ of a crown.

33. What is the cube of that number which, when multiplied by $\frac{3}{5}$ of $\frac{4}{5}$ of $1\frac{1}{3}$, will produce 1?

34. What is the net annual income of a man whose estate is worth £1023 10s. per annum, but who pays as land tax 2s. $8\frac{1}{2}$d. in the pound?

35. If the owner of $\frac{4}{17}$ of a ship sold $\frac{3}{11}$ of $\frac{2}{3}$ of his share for £$\frac{400}{33}$, what was the value of $\dfrac{1\frac{1}{4}}{4\frac{5}{8}}$ of $\frac{2}{5}$ of the whole ship at the same rate?

36. If two-thirds of a business be worth £440, what is the value of $\frac{3}{11}$ of it?

37. If $\frac{7}{15}$ of a guinea be taken from $\frac{3}{13}$ of $\frac{8}{9}$ of a five pound note, what fraction of £100 will remain?

38. A owns $\frac{9}{16}$ of a field, and B the rest, the difference between their allotments is 3 roods 17 poles, what is the area of the field?

SECTION VIII.—DECIMAL FRACTIONS.

246. A Fraction is called Decimal when its denominator is either ten, one hundred, one thousand, or some power of ten.

Thus, $\frac{3}{10}, \frac{5}{100}, \frac{7}{1000}, \frac{183}{100000}$ are Decimal Fractions.

Two or three facts concerning the principle of our Notation require to be distinctly remembered here.

I. In Integral Arithmetic the number 10 is the uniform instrument of multiplication, and we deal with all the collection of numbers which are brought before us as composed of tens and of powers of ten. Thus, if we have to think of $6 + 9 + 8$ we instantly resolve it into 23, or *two* TENS and 3; or if the number 5 times 7 is spoken of, we do not deal with it in arithmetic until it is transformed into 35, *i.e.*, *three tens* and 5.

II. Every figure in the following scale means 10 times more than that on its right; the last figure alone meaning 7, and all the rest having a higher value.

7 7 7 7 7

Here the first 7 signifies 7 times the fourth power of 10, the second 7×10^3, the third $= 7 \times 10^2$, and the fourth 7×10.

247. It will thus be seen that the object of all Integral Arithmetic is to express every collection of numbers whatever, their sums, their differences, their products, or their quotients, *in the form of multiples either of* 10 *or of some power of* 10. So that in fact 10 or some power of 10 is the multiplier always understood, though not expressed, as belonging to every figure in a line except the last.

248. It is the object of Decimal Fractions to express all *fractions* whatever, *i.e.*, all divisions of units into parts, as divisions by ten or by some power of ten.

249. *Observation.*—If we have to express 9 sevens we write down the multiplier thus, 9×7. But to express 9 tens it would not be necessary to write the 10, for by placing the 9 with one figure to its right, as 90 or 96, the multiplication is at once effected. So also if we mean $\frac{7}{9}$, or 7 divided by 9, we have to write down the divisor 9; but if we wish to deal with $\frac{7}{10}$, or 7 divided by 10, the decimal system enables us to express this without writing the 10.

250. In (246) each of the figures 77777 represented a tenth part of that on its left. In Fractions we can extend this arrangement below unity, still letting each figure mean a tenth of that on its left. In the following line let the figure with the line above it mean units.

$$\overset{5}{7}\ \overset{4}{7}\ \overset{3}{7}\ \overset{2}{7}\ \overset{1}{7}\ \overset{-}{7}\ \overset{1}{7}\ \overset{2}{7}\ \overset{3}{7}\ \overset{4}{7}\ \overset{5}{7}$$

Then the figure to its left is 7×10 or 70, but that on its right is $\frac{7}{10}$ or $7 \div 10$. So also the figure in the second place to the left of the unit $= 700$ or 7×10^2, but that two places to the right $= \frac{7}{100}$ or $7 \div 10^2$. The next pair of corresponding figures to the left and right mean respectively 7000 or 7×10^3, and $\frac{7}{1000}$ or $7 \div 10^3$. The next two are 70000 or 7×10^4, and $\frac{7}{10000}$ or $7 \div 10^4$, &c., &c.

Here the middle figure alone refers to unity, and not the last, as is the case in integer numbers, yet the same principle of notation still applies, viz., *that the value of the numbers increases* 10 *times at each place to the left*, and that each figure represents $\frac{1}{10}$ of the value it would have one step further to the left.

251. Instead of placing a mark over the unit as we have done, it is usual to place a point (.) between the unit and the first fractional number, thus:—

hundred thousands.	ten thousands.	thousands.	hundreds.	tens.	units.		tenths.	hundredths.	thousandths.	ten thousandths.	hundred thousandths.
5	6	8	3	0	4	·	2	4	7	3	9

$$578{\cdot}324 = (5 \times 100) + (7 \times 10) + 8 + \tfrac{3}{10} + \tfrac{2}{100} + \tfrac{4}{1000}.$$
$$70{\cdot}069372 = (7 \times 10) + \tfrac{6}{100} + \tfrac{9}{1000} + \tfrac{3}{10000} + \tfrac{7}{100000} + \tfrac{2}{1000000}.$$

Exercise LXXIII.

Write out the separate value of each figure in the following lines:—

1. 2970·32604; 3·087.
2. 14·109; ·604.
3. 1·684; ·052.
4. 103·561; 75·1086.
5. 3271·0986; 176·987.
6. 12·3456; 1·23456.
7. 507·1862; 90·006.
8. 12·037; 713·586.

252. *Observation.*—In writing integer numbers (11) there is always one place (that on the right) reserved for the unit, and if there is no unit the place is filled by a cipher (0). But in writing decimal fractions it is not necessary to reserve a place for the unit, as the first figure to the right of the decimal point always means tenths. Thus the two arrangements do not appear at first sight as if they corresponded exactly, for in integers the *second* figure in the series means tens, while in fractions the *first* figure means tenths; so also the third figure to the left of the point means hundreds, while the *second* figure to its right means hundredths. This will be seen in the two cases—

3640 and ·463.

In the former, begining from the unit, we find $4 \times 10, 6 \times 10^2$, and 3×10^3; in the latter, beginning with the unit, we have $\frac{4}{10}$, $\frac{6}{100}$, and $\frac{3}{1000}$. So that although the two expressions correspond exactly in meaning, the one being above unity and formed by decimal multiplication, and the other being below unity and formed by decimal division; yet 4 figures are needed to express the first, and only 3 to express the second. Nevertheless it must be remembered that in all expressions which are partly integral and partly fractional, the figures at equal distances from the unit correspond in value, thus:—

7 3 2 5 . 6 4 9

Here the 2 and the 6, which are equally distant from 5, correspond in value, the first meaning tens and the second tenths. So the 3 means three hundreds and the 4 four hundredths, each being in the third place from the unit. The language employed in (15) will therefore still be available, tens and tenths being both units of the second place, thousands and thousandths being both units of the fourth place, millions and millionths being both units of the seventh place. When the fraction stands alone without an integer the point itself must be counted as one, and the first figure (tenths) as being a unit of the second place, and then the language of Integral Notation will still apply.

253. *In Integer numbers a cipher to the right of a figure increases its value ten times; but a cipher to its left leaves it unaltered. In Fractional numbers a cipher to the left of a number diminishes its value ten times, but a cipher to the right leaves it unaltered.*

254. The cipher (16) is simply intended to put any given figure in its right place with regard to the unit. In integer numbers, if a cipher does not stand to the right of a figure it does not affect its value at all. Thus, 057 means the same as 57, the 0 has no meaning, for it does not stand between any figure and the unit; but if

the number were 2057 the 0 would affect the value of the 2 by placing it a step further from the unit, although it would not affect the value of the 5 or the 7. Similarly, ciphers have a meaning in Decimal Fractions when they place a number further from the unit, but not otherwise. In the expression ·57, the 5 means $\frac{5}{10}$ and the 7 $\frac{7}{100}$; but in the expression ·5700 they still retain the same value, and the ciphers, because they do not affect the position of either the 5 or the 7 with regard to the unit, are without meaning. But in ·057 the cipher really affects the value of both, for it makes the 5 mean $\frac{5}{100}$ instead of $\frac{5}{10}$, and the 7 $\frac{7}{1000}$ instead of $\frac{7}{100}$.

255. *Corollary.—In any line of figures, to move the decimal point one place to the right is to multiply the whole line by* 10, *but to remove the point one place to the left is to divide the whole line by* 10. *To shift the point two places is to multiply or divide by a hundred; three places by a thousand, &c.*

Example.—(a). 1783·964. (b). 178·3964. (c). 17839·64.

In (a) the figure 3 is the unit, and every figure is greater or less according to its distance from that unit; but in (b) the decimal point having been removed one place to the left, the 8 which meant 8 tens becomes 8 only; the 3 which meant 3 units becomes the 10th part of 3, and every figure in (b) represents a 10th part of the value it had in (a). Now, on comparing (a) and (c), we find that the opposite effect has been produced by putting the point one place to the right. For in (c) the 3 means 3 tens instead of 3 units; the 8 means 8 hundreds instead of 8 tens, and every figure means ten times what it means in (a). By comparing (b) and (c) we find that every figure in the latter means a hundred times its value in the former, because the decimal point is two places further to the right.

Exercise LXXIV.

Multiply the following expressions—

1. By ten. 1·078; 56·320; ·079; 513·087; 16·54.
2. By a hundred. 235·96; 50·985; ·10372; ·001; 1·8739.
3. By ten thousand. 41·0968; 1·50796; ·00183; 723·568701.

Divide the following numbers—

1. By ten. 35·607; ·178; 4·762; ·0018; 3·729.
2. By a thousand. 13·569; 27·48; 47·2; 3·069; 18372·1.
3. *By a million.* 27486·32; 5098·527; 18·07629; 453786·1.

256. Decimals have this advantage over vulgar fractions, that they can be instantly reduced to a common denominator, and can therefore be added, subtracted, or compared without trouble.

Example.—The figures 792·8345 taken separately represent

$$(7 \times 10^2) + (9 \times 10) + 2 + \tfrac{8}{10} + \tfrac{3}{100} + \tfrac{4}{1000} + \tfrac{5}{10000}.$$

10000 is here the greatest denominator in the series, and is also a common multiple of all the other denominators. Take the fraction $\frac{4}{1000}$, multiply its terms by 10 and it becomes $\frac{40}{10000}$; the former fraction, in like manner, becomes $\frac{300}{10000}$; $\frac{8}{10}$ becomes $\frac{8000}{10000}$, and so on.

Thus $700 = \frac{7000000}{10000}$, $90 = \frac{900000}{10000}$, $2 = \frac{20000}{10000}$, $\frac{8}{10} = \frac{8000}{10000}$, $\frac{3}{100} = \frac{300}{10000}$, $\frac{4}{1000} = \frac{40}{10000}$, and $\frac{5}{10000}$.

$$792{\cdot}8345 = \frac{7000000 + 900000 + 20000 + 8000 + 300 + 40 + 5}{10000} = \frac{7928345}{10000}.$$

A separate analysis of the parts of this expression gives the following results:—

$$792{\cdot}8345 = 700 + 90 + 2 + \tfrac{8}{10} + \tfrac{3}{100} + \tfrac{4}{1000} + \tfrac{5}{10000}.$$
$$792{\cdot}8345 = 792 + \tfrac{8}{10} + \tfrac{3}{100} + \tfrac{4}{1000} + \tfrac{5}{10000}.$$
$$792{\cdot}8345 = \tfrac{7928}{10} + \tfrac{3}{100} + \tfrac{4}{1000} + \tfrac{5}{10000}.$$
$$792{\cdot}8345 = \tfrac{79283}{100} + \tfrac{4}{1000} + \tfrac{5}{10000}.$$
$$792{\cdot}8345 = \tfrac{792834}{1000} + \tfrac{5}{10000}.$$
$$792{\cdot}8345 = \tfrac{7928345}{10000}.$$

Exercise LXXV.

Decompose each of the following fractions into four equivalent expressions, as in the example:—

1.	213·5;	40·687.	2.	53·089;	2790·387.
3.	42·5068;	1209·385.	4.	107·98;	·10798.
5.	6305·792;	270·69.	6.	41·372;	809·6274.
7.	61·08;	5·079.	7.	32·765;	10·9721.

257. Hence every line of figures having a decimal point may be considered the numerator of a fraction, whose denominator is 10 raised to the power indicated by the number of figures to the right of the decimal point.

Example.—$\cdot172 = \frac{172}{10^3} = \frac{172}{1000}$; $2053 = \frac{2053}{10^2}$ or $20\frac{53}{100}$.
$4{\cdot}7296 = 4\frac{7296}{10^4}$ or $\frac{47296}{10000}$; $.0083 = \frac{83}{10^4} = \frac{83}{10000}$.

258. TO REDUCE A DECIMAL TO THE FORM OF A VULGAR FRACTION—

RULE.

Place the whole of the figures as the numerator of the fraction, omitting the decimal point, and place as the denominator the figure 1, followed by as many ciphers as there are figures to the right of the decimal point.

EXERCISE LXXVI.

Reduce the following decimals into equivalent vulgar fractions.

(When whole numbers occur represent the decimals in two ways, (a). As mixed numbers; (b). As improper fractions.)

1. 235·79; ·0186; 5·072.	2. 8·07; ·123; 1·23.
3. 40·327; 5·69; 247·85.	4. ·005; 6·078; 17·28.
5. 27296·8; 807·324; ·307.	6. 7·129; 18·736; 4·7293.
7. 8·072; 370·296; 10·72.	8. ·0001; 100·7; 308·6.

259. Vulgar fractions cannot always be converted into decimals which are exactly equivalent; but we may always obtain as much accuracy as we wish in the decimal form. There are two methods of effecting this change.

260. *I. Method of equal multiplications and divisions.*

Example I.—To reduce $\frac{3}{4}$ to a decimal form is to find a fraction which shall equal $\frac{3}{4}$ and yet shall have 10 or some power of 10 for its denominator. Multiply both terms by 100; then $\frac{3}{4} = \frac{300}{400}$. Divide both by 4; then $\frac{300 \div 4}{400 \div 4} = \frac{75}{100} = {\cdot}75$, which is the decimal equivalent to $\frac{3}{4}$.

Example II.—To convert $\frac{5}{7}$ into a decimal form: multiplying numerator and denominator by 10,000 the fraction becomes $\frac{50000}{70000}$; dividing both terms by 7, we have this result— $7\left|\frac{50000}{70000}\right. = \frac{7142\frac{6}{7}}{10000}$; neglecting the remainder, we have here a fraction $\frac{7142}{10000}$ in a decimal form, and expressible thus ·7142. It is not exactly equal to $\frac{5}{7}$, but it does not differ from it by so much as $\frac{1}{10000}$, and it is evident that if we had extended the same process to millionths we should have arrived at a still closer approximation to its value. Thus—

$\frac{5}{7} = \frac{5000000}{7000000}$; dividing both by 7 $\left| \frac{5000000}{7000000} = \frac{714285\frac{5}{7}}{1000000} = \cdot 714285 \right.$

Here the decimal expression ·714285, though still not exactly equivalent to $\frac{5}{7}$, does not differ from it by so much as a millionth of a unit, and may therefore be considered practically as its equivalent. There is no limit to the extent to which this process may be carried, and therefore the error may be reduced to as small an amount as we please.

261. *II. Method of reduction.* From (103) it will be seen that the division of all quantities is effected by reducing the remainders step by step into equivalent numbers of a lower name; the same method is applicable here.

Example.—Let it be required to reduce $\frac{5}{19}$ to a decimal fraction, *i. e.*, to divide 5 by 19 in such a way that the answer shall appear in the form of tenths, hundredths, thousandths, &c. For this purpose we will reduce these 5 whole numbers into the required parts.

```
19) 5 (0 whole numbers
   10
19) 50 (2 tenths
    38
    12
    10
19) 120 (6 hundredths
    114
      6
     10
19) 60 (3 thousandths
    57
     3
    10
19) 30 (1 ten thousandth
    19
    11
    10
19) 110 (5 hundred thousandths
     95
     15
     10
19) 150 (7 millionths
    133
     17
```

Here, finding that the 19th part of 5 gives no answer in integers, we reduce the 5 into tenths, and find that the 19th part of 50 tenths gives 2 tenths and 12 tenths over. These 12 tenths are reduced into 120 hundredths, and the 19th part of them is 6 hundredths, with a remainder 6 hundredths. These 6 hundredths are again reduced into the next lower name, and divided by 19; and all the remainders will be seen to have been treated exactly as in compound division. The answer, carried as far as millionths, appears to be, $\frac{2}{10} + \frac{6}{100} + \frac{3}{1000} + \frac{1}{10000} + \frac{5}{100000} + \frac{7}{1000000} = \frac{263157}{1000000} = \cdot 263157.$

Now as reduction into the next power of 10 may simply be effected by adding a cipher, it is evident that we have employed more

19)50(263157
  120
   60
    30
   110
    150
     17
·263157 Ans.

figures in this example than are necessary. The example in the margin shows all the figures which are necessary (102) to obtain the result, and it is evident that we may go on adding ciphers to the remainder, or reducing them to lower names, as long as we please, every step giving us a nearer decimal approximation to the value of $\frac{5}{19}$.

262. TO REDUCE VULGAR FRACTIONS TO A DECIMAL FORM—

RULE.

Divide the numerator by the denominator. If any quotient arises it is an integer number; but if not, or if there be a remainder, add a cipher, and continue the division until there is no remainder.

263. *Observation.*—The quotient obtained after adding one cipher stands in the place of tenths (*i. e.*, next after the decimal point); the quotient obtained after adding two ciphers stands in the second place; after adding 3 ciphers in the third place; 4 ciphers in the fourth place, &c.

EXERCISE LXXVII.

Reduce the following fractional expressions to decimals true to the fifth place:—

1. $\frac{4}{7}$; $\frac{519}{284}$; $\frac{31}{29}$.
2. $\frac{2478}{5196}$; $\frac{509}{625}$; $\frac{8}{9}$.
3. $\frac{7}{12}$; $\frac{54}{75}$; $\frac{68}{34}$.
4. $\frac{59}{64}$; $\frac{21}{125}$; $\frac{16}{18}$.
5. $\frac{17}{16}$; $\frac{27}{200}$; $\frac{34}{215}$.
6. $\frac{24}{150}$; $\frac{276}{345}$: $\frac{18}{50}$.
7. $2\frac{7}{8}$; $13\frac{2}{5}$; $122\frac{7}{9}$.
8. $\frac{9}{17}$; $\frac{232}{546}$; $\frac{183}{1794}$.
9. $\frac{180}{576}$; $12\frac{3}{4}$; $2\frac{4}{7}$.
10. $\frac{2}{7}$ of $\frac{5}{8}$; $\frac{4}{5}$ of $\frac{18}{19}$.
11. $\dfrac{\frac{2}{7} \times \frac{5}{17}}{\frac{3}{4} + \frac{2}{3}}$; $\dfrac{18\frac{2}{7} - 3\frac{5}{8}}{16\frac{8}{9}}$.
12. $\dfrac{\frac{4}{5} \text{ of } 7\frac{1}{2}}{\frac{3}{4} \text{ of } 19\frac{2}{6}}$; $\dfrac{\frac{1}{2} + \frac{1}{3} + \frac{1}{4}}{\frac{1}{2} \text{ of } \frac{1}{3} \text{ of } \frac{1}{4}}$.

RECURRING OR CIRCULATING DECIMALS.

264. In many cases it has been seen that however far we carry the answer there is still a remainder; that is to say, it is not possible to represent the given fraction exactly in a decimal form. Sometimes it may be seen at once after the operation is begun, that the same remainder occurs a second time, and that consequently the same set of figures will recur in the quotient. Thus—

$$\frac{1}{3} = \frac{10000 \div 3}{30000 \div 3} = \frac{3333}{10000} = \cdot 3333.$$

For as the same remainder occurs every time, the same quotient will recur *ad infinitum*. This is called a recurring or interminate decimal.

Sometimes also a series of different figures recurs in the same order. On reducing $\frac{2}{7}$ to a decimal we find the following result:—

$$\begin{array}{l} 7)\underline{20} \\ 285714\ 285714 \cdots \end{array}$$

Here, after we have passed the sixth figure of the quotient, we have the remainder 2; but as this is the figure with which we started, we shall, of course, the divisor remaining the same, obtain the same set of figures in the quotient again. The decimal $\cdot\dot{2}8571\dot{4}$ is therefore called a repeating or circulating decimal.

When, as in these cases, the same figures recur from the beginning, the expressions are called *Pure Circulating Decimals*. But the fraction $\frac{7}{12}$, for example, is found to equal $\cdot 5833\dot{3}$, &c. The first two figures, 58, do not repeat, but the 3 does. Such an expression is called a *Mixed Circulating Decimal.*

The figures which are repeated are called the "repetend." When only one figure recurs it is called a "simple repetend;" when more than one, they form what is called a "compound repetend." Thus in the circulating decimal ·333, &c., 3 is a simple repetend. In ·962962962, &c., 962 is a compound repetend.

It is usual to indicate a circulating decimal by placing a point over the first and last of the recurring digits. Thus—

$$\begin{array}{lcl} \cdot\dot{6} & = & \cdot 666 \cdots \\ \cdot\dot{2}\dot{7} & = & \cdot 272727, \text{\&c.} \\ \cdot 58\dot{3} & = & \cdot 58333, \text{\&c.} \\ \cdot\dot{2}9\dot{6} & = & \cdot 296296296, \text{\&c.} \end{array}$$

265. *Every vulgar fraction whose value cannot be exactly expressed decimally, will take the form of a circulating decimal.*

For as every remainder must be less than the divisor, the number of remainders which can occur is limited; and because a cipher is added to every remainder, when any dividend occurs a second time, the same set of dividends and quotients must appear as before.

Exercise LXXVIII.

Reduce the following fractions to circulating decimals:—

1. $\frac{7}{11}$; $\frac{5}{9}$; $\frac{23}{21}$. 2. $\frac{4}{7}$; $\frac{7}{12}$; $\frac{8}{19}$. 3. $\frac{5}{13}$; $\frac{6}{17}$; $\frac{4}{15}$.
4. $\frac{17}{23}$; $\frac{12}{19}$; $\frac{6}{27}$. 5. $\frac{8}{17}$; $\frac{5}{13}$; $\frac{3}{7}$. 6. $\frac{12}{18}$; $\frac{2}{3}$; $\frac{8}{9}$.

266. It must be observed that, in every repeating decimal, the whole value of the quantity represented is not expressed; for as the series is infinite we may carry it as far as we will, and yet some part of the fraction will remain unwritten. These parts may be very small, yet they are truly parts of the fraction, and in the decimal system we are obliged to neglect them. The following is an easy method for finding a vulgar fraction, which represents the exact value of a pure circulating decimal; and it will be seen that, whereas by prolonging the decimal line, we continually approach, but never reach, the true expression of its value; it is always possible to give an accurate expression in the form of a vulgar fraction.

Example I.—Let S represent the sum of the series ·4444· · ·

$$\begin{array}{lrcl} \text{(a). Then (255)} & 10\,S & = & 4{\cdot}4444\cdots\cdots \\ \text{(b). But} & S & = & {\cdot}4444\cdots\cdots \\ \hline \text{Subtracting (b) from (a)} & 9\,S & = & 4 \quad \therefore \quad S = \tfrac{4}{9}. \end{array}$$

Here, as it is manifest that the indefinite expression ·444· · · · means the same in both lines, the one may be subtracted from the other, and will leave no remainder. But as $10\,S - S = 9\,S$; it follows that 4 equals 9 times the required value, which is therefore $\frac{4}{9}$.

Example II.—Let the circulating decimal be ·252525; then because there are two figures in the repetend it will be convenient to multiply the sum by 100. Then as before, let $S = {\cdot}\dot{2}\dot{5}$—

$$\begin{array}{lrcl} \text{Then} & 100\,S & = & 25{\cdot}252525, \&c. \\ \text{Subtract} & S & = & {\cdot}252525, \&c. \\ \hline & 99\,S & = & 25. \quad \text{Therefore } S = \tfrac{25}{99}. \end{array}$$

Example III.—Let the recurring decimal be ·142857; and as its denominator is a million, it will be convenient to multiply by that number. Here as before, let $S = {\cdot}\dot{1}4285\dot{7}$—

$$\begin{array}{lrcl} \text{Then} & 1000000\,S & = & 142857{\cdot}\dot{1}4285\dot{7}, \&c. \\ \text{Subtract} & S & = & {\cdot}\dot{1}4285\dot{7}, \&c. \\ \hline & 999999\,S & = & 142857. \quad \text{And } S = \tfrac{142857}{999999}. \end{array}$$

267. When Mixed Circulating Decimals occur they should be treated as follows:—

I. Reduce $\cdot 5833\dot{3}$ to a vulgar fraction.

$$\begin{array}{lrcl} \text{Multiply by 1000} \quad \text{Then} & 1000\,S & = & 583{\cdot}3\dot{3} \text{ (a)} \\ \text{Multiply by } 100 & 100\,S & = & 58{\cdot}33 \text{ (b)} \\ \hline \text{Subtracting (b) from (a)} & 900\,S & = & 525 \quad \therefore \; S = \tfrac{525}{900}. \end{array}$$

II. Reduce $\cdot 94\dot{7}2\dot{4}$ to a vulgar fraction.

Multiply by 100000	Then 100000 S	=	$94724\cdot\dot{7}2\dot{4}$
Multiply by 100	Then 100 S	=	$94\cdot 724$
Subtract	Then 99900 S	=	94630
	$\therefore$ S	=	$\frac{94630}{99900}$.

In each of these cases we first multiply by a power of 10, high enough to make an integer number of the recurring and the non-recurring parts; we then multiply the expression by a power of 10, high enough only to make a whole number of the part which does not recur: on subtracting this from the other, the circulating decimal disappears, and the whole number which results has for its denominator the difference between the greater power of ten and the less. In the former case one tenth part, and in the latter one hundredth part, has been subtracted from both numerator and denominator.

268. To reduce a circulating decimal to a vulgar fraction—

If only the same figures recur, make the repetend the numerator of a fraction, and place underneath it as many nines as there are digits in the repetend.

But if it be a mixed circulating decimal, subtract the digits which do not recur from the whole expression as far as the end of the first repetend; take this difference as the numerator, and for the denominator place as many nines as there are in the recurring part, followed by as many ciphers as are in the part which does not recur.

Exercise LXXIX.

Find vulgar fractions equivalent to the following expressions :—

1. $\cdot\dot{4}\dot{7}$; $\cdot\dot{5}$; $\cdot 0\dot{5}$. 2. $\dot{3}2\dot{7}$; $\cdot\dot{9}523\dot{8}$; $\cdot\dot{7}1428\dot{5}$.

3. $\cdot 00\dot{1}8\dot{5}$; $\cdot 3\dot{1}3\dot{2}$; $1\cdot 70\dot{1}\dot{6}$. 4. $\cdot 28\dot{1}7\dot{2}$; $3\cdot 4\dot{1}\dot{5}$; $\cdot 70\dot{2}3\dot{8}$.

5. $6\cdot\dot{0}3\dot{8}$; $\cdot 71\dot{2}\dot{5}$; $\cdot 80\dot{5}6\dot{3}$. 6. $\cdot 13\dot{5}8\dot{7}$; $\dot{7}1\dot{9}$; $6\cdot 0\dot{2}\dot{7}$.

7. $1\cdot 27\dot{3}1\dot{6}$; $45\cdot 5\dot{0}\dot{8}$; $\cdot\dot{8}2\dot{7}$. 8. $\cdot 7\dot{0}3\dot{2}$; $6\cdot 4\dot{1}$; $7\cdot 0\dot{4}\dot{3}$.

9. $27\cdot 0\dot{5}\dot{6}$; $3\cdot 00\dot{7}\dot{5}$; $4\cdot\dot{1}2\dot{7}$. 10. $11\cdot\dot{3}7\dot{2}$; $4\cdot 16\dot{8}$; $\cdot 5\dot{0}93\dot{2}$.

11. *What is the difference between* $\cdot\dot{0}\dot{7}$ *and* $\cdot 0\dot{7}$?

269. ***No vulgar fraction can be precisely expressed as a decimal if its denominator can be resolved into any other prime factors than 2 and 5.***

Demonstrative Example.—The fraction $\frac{7}{9}$ cannot be reduced lower because its terms are prime, but (196) any fraction can be altered into the form of one of a higher name, provided that the new denominator is a multiple of the former. If either 10, or 100, or 1000, or any power of 10 be also a multiple of 9, the fraction $\frac{7}{9}$ can be reduced to its exact equivalent in the decimal system, but if not, the fraction must take the form of a repeating decimal.

The question to solve in this case is, "Can any multiple whatever of the number 9 be found equal to any power of ten?" If it can, then 9 must be a measure of either 10 or some power of ten. But it may be inferred from (168) that if a number measure another the second must contain all the prime factors of the first. Now the only prime factors of 10 and of all the powers of 10 are 2 and 5, and neither of these is a factor of 9; wherefore no multiple of 9 can ever equal a number whose prime factors are 2 and 5. In the same manner it might be proved that whenever the denominator of a vulgar fraction, expressed in its lowest form, is resolved into any other prime factors than 2 or 5, the fraction will be interminate.

270. TO DETERMINE WHETHER A VULGAR FRACTION CAN BE EXPRESSED DECIMALLY—

RULE.

Reduce the vulgar fraction to its lowest terms (198). Find the prime factors of the denominator (159). If they are any others than 2 or 5 the fraction will take the form of a repeating decimal.

Example.—$\frac{7}{32}$. Here $32 = 2 \times 2 \times 2 \times 2 \times 2$. It has no other prime factor than 2: the fraction may therefore be expressed decimally. But in $\frac{8}{15}$, because $15 = 3 \times 5$, and no power of 10 has 3 for a prime factor, therefore the fraction is interminate.

EXERCISE LXXX.

State which of the fractions in Exercise LVIII. are capable of being expressed as decimals.

271. When fractions are extended to any great length, it becomes very cumbersome to work with them. The figures which stand in the 5th, 6th, and later places, represent exceedingly small values, are only needed in calculations of great nicety, and are often neglected. Whenever this is done it is necessary to look at the first figure which is neglected, and to attend to the following considerations :—

Suppose the fraction 17·168327 is given, and we only wish to deal with it as far as the second place of decimals; neglecting the four figures to the right we call it 17·16. Now the 8, which is the first of the digits cut off, means $\frac{8}{1000}$, and if $\frac{2}{1000}$ were added to it would become $\frac{10}{1000}$ or $\frac{1}{100}$, which is a number suitable to be carried into the next place to the left. When therefore I write the fraction as 17·16 the expression differs by $\frac{8}{1000}$ from the truth, but if I had written 17·17 it would only have differed by $\frac{2}{1000}$ from the truth. It is evident that in this case we make a greater error by omitting it than by adding one to the place on its left. Hence 17·17 is nearer to 17·168 than 17·16 is. But as the next figure to the right is 3, we should have to add $\frac{7}{10000}$ to it to make the fraction 17·169, whereas by leaving it out, and simply writing 17·168, we only depart $\frac{3}{10000}$ from the truth.

If 5 were the number, the error of wholly omitting it would evidently be the same as that of adding enough to it to make 1 in the next place to the left. Thus—

·237 true to 2 places = ·24, because 37 is nearer to 40 than to 30.

But 7·2835 expressed to 2 places is only 7·28, because 835 is nearer to 800 than to 900.

272. To find the nearest equivalent to any fraction as far as to any given place of decimals—

RULE.

If the first of the neglected figures is 5 or more than 5, add 1 to the last of the digits which are retained. But if the first of the rejected figures be less than 5, the former figures may be set down without them.

Exercise LXXXI.

Find the nearest equivalent to the decimals in Exercise LXXVI. as far as two decimal places.

SECTION IX.—ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF DECIMAL FRACTIONS.

Addition.

273. From (199) it appears that when fractions are reduced to a common denominator the numerators alone may be added or subtracted to give the sum or difference of any two fractions. But in Decimals the numerators only of the fractions are expressed, and the denominators, which are understood, are easily made common by the mere arrangement of the figures in a certain order.

Suppose it is required to find the sum of the expressions—

$$7{\cdot}2916 + 80{\cdot}25 + 423{\cdot}41 + 8071{\cdot}2 + 7056{\cdot}8103.$$

Then,

$$\begin{aligned}
7{\cdot}2916 &= & &\tfrac{72916}{10000} \\
80{\cdot}2500 &= \tfrac{8025}{100} &= &\tfrac{802500}{10000} \\
423{\cdot}4100 &= \tfrac{42341}{100} &= &\tfrac{4234100}{10000} \\
8071{\cdot}2000 &= \tfrac{80712}{10} &= &\tfrac{80712000}{10000} \\
7056{\cdot}8103 &= & &\tfrac{70568103}{10000} \\
\hline
17638{\cdot}9619 &= & &\tfrac{176389619}{10000}
\end{aligned}$$

Here ciphers have been added in order to reduce all the fractions to the common denominator 10000; and it is evident that, neglecting the consideration of the decimal point altogether, every line of figures may be regarded as an integer number, the numerator of a fraction, of which the unexpressed denominator is 10000. On adding these numerators together, as whole numbers, the sum is found to be 176389619, but these figures give the numerator of the sum of the fractions; and as the common denominator is 10000, or the fourth power of 10, the point must be placed four figures from the right, and the answer is 17638·9619.

The ciphers which have been introduced are not necessary, as the value of each figure is sufficiently known by its position.

274. The reason of the ordinary rule may be further evident from another example.

Add together *·0018, 7·96, 413·587, 21·409.*

·0018
7·96
413·587
21·409
442·9578

Here the figure standing furthest to the right is 8, and means $\frac{8}{10000}$; there are no other figures of that value, so we set down 8 in the fourth place from the unit. In the next column are 9, 7, and 1, or 17; these are $\frac{17}{1000}$, but $\frac{17}{1000} = \frac{10}{1000} + \frac{7}{1000} = \frac{1}{100} + \frac{7}{1000}$, so the 7 are put in the thousandths place, and $\frac{10}{1000}$, or $\frac{1}{100}$, are carried into the hundredths place as 1. This 1 added to 8 and 6 makes 15, and these are 15 hundredths; but $\frac{15}{100} = \frac{1}{10} + \frac{5}{100}$, we therefore set down the 5 as hundredths, and carry 1 to the tenths. $\frac{1}{10} + \frac{4}{10} + \frac{5}{10} + \frac{9}{10} = \frac{19}{10}$, but $\frac{19}{10} = 1 + \frac{9}{10}$, we therefore set down the 9 among the tenths and carry the 1 to the other side of the decimal point as a whole number. The rest is the ordinary Addition of Integers.

275. To add decimal fractions—

RULE.

Arrange the figures so that the decimal points in all the lines shall fall in one vertical column. Add up as in Simple Addition, placing a decimal point in the answer exactly underneath the other points.

EXERCISE LXXXII.

1. 279·806 + 304·72 + ·008 + ·596 + 2·037.
2. 108·62 + ·001 + ·1007 + 3·8 + 173·6 + 17·365.
3. 58·72 + 96·057 + 41·2874 + 3·027 + 1865·07.
4. 71·9683 + 2·17 + 4·621 + ·008 + 72·0963 + ·04.
5. 632·1874 + 4·017 + 163·5 + 8·047 + 419·8 + 3·74.
6. 71·827 + 3·142 + 6·1547 + 103·03 + 51·0008.
7. 820·96 + 70·03 + ·008 + 10·72 + 13·5678.
8. 927·416 + 8·274 + 372·6 + 62·07938 + ·507462.
9. 103·72 + 11·7 + 61·187 + 3·0972 + 2·073 + 864·145.
10. 27·07 + 83·6 + 409·6 + 3·1725 + 8·627 + ·0072.
11. 2·124 + 8·327 + 65·47 + ·2198 + 327·4 + 862·1.
12. 57·213 + 8627·9 + 4138·7 + 65·41 + ·00728 + 1·05.
13. Add together 17 thousandths, 2 tenths, and 47 millionths.
14. Add together 238 tenths, 453 thousandths, 6134 millionths, and 18 ten thousandths.
15. Find the sum of 264 hundredths, 18 tenths, 34 millionths, and 62584 hundred thousandths.

SUBTRACTION.

276. This Rule, like Addition, is always a simple process when the fractions have a common denominator. Thus, if it be required to take 7·314 from 25·06, we have (257) two vulgar fractions, $\frac{7314}{1000}$ and $\frac{2506}{100}$, of which the latter, $\frac{2506}{100}$, can be brought to the same name as the former by adding a cipher to the numerator and denominator. It then becomes $\frac{25060}{1000}$. We have now to find the difference between $\frac{7314}{1000}$ and $\frac{25060}{1000}$. Taking (210) the difference of the numerators only we find—

$$\frac{25060}{1000} - \frac{7314}{1000} = \frac{17746}{1000} = 17·746.$$

Or, 25·060 7·314 17·746	Here we simply place the figures of like values under one another, assume a cipher in the upper place on the right, and proceed as in Simple Subtraction.

277. *Observation.*—On examining sums of this kind more closely we shall find that what was said of Simple Subtraction is equally true here. We do not actually subtract the required quantity from the other, but in nearly all cases we add something to both; and we take the subtrahend + this added number, from the minuend + the same number. Thus in the sum, subtract 20·758 from 301·2439,

	hundreds.	tens.	units.	tenths.	hundredths.	thousandths.	ten thousths.
301·2439	3	10	1	12	14	13	9
20·758	1	2	1	8	6	8	
280·4859 =	2	8	0	·4	8	5	9

instead of 3, in the thousandths place of the minuend, we have taken 13; but we have also turned the 5, of the hundredths place of the subtrahend, into 6. Thus to the upper line we have added $\frac{10}{1000}$, and to the lower $\frac{1}{100}$. But these are equal, therefore by (44) they do not affect the answer. So also we have added $\frac{10}{100}$ to the 4 in the upper line, and $\frac{1}{10}$, which is the same as $\frac{10}{100}$, to the 7 of the lower line. $\frac{10}{10}$ have been added to the tenths of the upper line, and 1, which is equal to $\frac{10}{10}$, has been added in the units place of the lower line. In like manner 10 tens have been added to the minuend, and 1 hundred to the subtrahend. Thus equal additions have been made to both lines, and the work really effected has been,

From	$301·2439 + 10 \text{ tens} + \frac{10}{10} + \frac{10}{100} + \frac{10}{1000}$
Take	$20·758 + 1 \text{ hundred} + 1 + \frac{1}{10} + \frac{1}{100}$
	280·4859

278. TO SUBTRACT ONE DECIMAL FRACTION FROM A GREATER—

RULE.

Arrange the numbers so that the points are in a vertical line, and that figures of the same value are in corresponding places. Subtract as in Simple Subtraction, and place a point in the answer underneath the other two.

EXERCISE LXXXIII.

Work the following sums in Subtraction:—

1. 709·63 − ·8514; 234·057 − 18·5.
2. 72·065 − 19·7234; 81·963 − 1·7.
3. 2107·5462 − 17·19382; 18 − ·18.
4. 150·7 − 1·507; 216·9 − ·83472.
5. 1201·6 − 43·598; 47·106 − 8·271.
6. 1 − ·001; 827·43 − 97·6387.
7. 2123·5 − ·0078; 62·97 − ·63845.
8. 2172·81 − 31·629; 41·78 − ·29643.
9. 872·1 − (·8721 + ·008); 378·69 − (20·07286 + 81·6).
10. 34·72 + ·1862 − 6·82; 27·8 + 619·7 − 327·9.
11. (5·028 + ·00973) − (6·704 − 2·38).
12. 3·7246 + 4·1 + ·097 − 742; $7{\cdot}1\dot{8} + 62{\cdot}\dot{3} - {\cdot}\dot{5}$.
13. $4{\cdot}46 - {\cdot}1\dot{9}\dot{7}$; 80·23 − 7·6453.
14. What is the difference between the sum of 33 millionths and 17 thousandths, and the sum of 53 hundredths and 274 tenths?
15. What is the difference between $\frac{3}{8}$ and 1·0084?
16. How much greater is the difference between $2\frac{1}{5}$ and 1·256 than the sum of ·05684 and ·556?
17. A person had ·825 of a ship, worth £10,000, after he had sold $\frac{5}{40}$; what part of the ship was his at first?
18. Say how much the product of $2\frac{5}{7}$ and $\frac{3}{4}$ exceeds the difference between $3\frac{2}{3}$ and 1·98046.
19. Which is the greater, and by how much, $\frac{2}{3}$ of $\frac{8}{9}$ of £50, or £8·642 + £18·0564?
20. The franc weighs 77·17 grains, of which 69·453 are pure silver; what is the weight of the alloy?

MULTIPLICATION.

279. From (214) it appears that to multiply two fractions together we multiply the numerators together for the numerator of the product, and the denominators together for the denominator of the product. This is the principle which applies also to decimals. Now it is very easy to multiply the numerators, which in Decimals are always expressed in figures; but as the denominators are not expressed, but are only indicated by the pointing, it requires some care to know what the denominator of the product is, and how its value can be accurately marked.

Let it be required to find the product of ·005 and 1·27. This question is equivalent to $\frac{5}{1000} \times \frac{127}{100}$. But by (219) the product of these two fractions is $\frac{635}{100000}$, which expressed decimally is ·00635. Here, because 5 is one numerator and 127 the other, $5 \times 127 =$ the numerator of the product. But as in ·005 the 5 stands in the third place from the unit, it represents 5 divided by the third power of 10. Again, because 1·27 means 127 divided by the second power of 10, the two denominators are 10^3 and 10^2. But (137) the product of these is 10^5. So, in like manner, the product of any powers of 10 is represented by adding the numbers which are the exponents of those powers. Thus (137) $10^2 \times 10^7 = 10^{2+7} = 10^9$. So also $10^4 \times 10^3 = 10^{4+3} = 10^7$. Now because the denominator of every decimal fraction is always that power of 10 which is indicated by the number of figures to the right of the decimal point (*e.g.*, $1{\cdot}719 = \frac{1719}{10^3}$, $\cdot04132 = \frac{4132}{10^5}$, &c.), it follows that by adding together the number of places on the right of the point in both factors, we learn how many places should be to the right of the point in the product. The following examples will illustrate this:—

1. $\cdot5 \times \cdot7 = \frac{5}{10} \times \frac{7}{10} = \frac{5 \times 7}{10 \times 10} = \frac{35}{100} = \cdot35.$
2. $\cdot004 \times 1{\cdot}7 = \frac{4}{10^3} \times \frac{17}{10} = \frac{68}{10^4} = \cdot0068.$
3. $5 \times \cdot00003 = 5 \times \frac{3}{10^5} = \frac{15}{10^5} = \cdot00015.$
4. Multiply 2780·961 by 1·32.

$$
\begin{array}{r l}
2780{\cdot}961 & = 2780961 \div 10^3 \\
1{\cdot}32 & = 132 \div 10^2 \\
\hline
5561922 & \\
8342883 & \\
2780961 & \\
\hline
3670{\cdot}86852 & = \dfrac{2780961 \times 132}{10^5}
\end{array}
$$

280. To multiply decimal fractions—

RULE.

Multiply the numbers as integers. Mark off as many places to the right of the decimal point in the product as there are in all the factors put together.

Exercise LXXXIV.

Find the product of the following decimal expressions:—

1. 27·98 × 63·5; 20·5 × 318·62.
2. 3·5 × 4·7 × 18·05; 16 × 27·1 × ·817.
3. 712·4 × 81·67 × 21; 2·03 × 203 × ·203.
4. 17·186 × ·5198; ·007 × 31·5 × ·617.
5. 61·58 × 317·2; 31·87 × 61·98 × 214·7.
6. 713·72 × 81·076 × 2·03; 8·145 × 614·7 × 30·03.
7. (2·7 + 31·85) × (3·16 − ·316); (4·198 − 1·7096) × 31·278.
8. (12·7 × 3·16 × 2·1) + (27·3 × 10·8); (61·95 + 87·2) − (21·3 × 6·19).
9. (71·42 × 3·164 × 21·008) − 17·45; ·002 × 2 × 65·7.
10. (107·8 + 6·541 − 31·96) × 1·742; 834·72 − (61·35 × 2·007).
11. 2178·6 × 18·74 × 21·72; (61·87 + 2·19 − 3·07) × 4·86.
12. Find the product of the sum and difference of 21·8 and ·324.
13. Work the last four sums in Exercise LXIII. decimally.
14. One brother has left him $\frac{7}{8}$ of a vessel, the whole of which is worth £8,000, while a second has for his portion 72 shares in the Great Western Railway, worth £75·125; whose property is the most valuable?
15. If in a certain town the number of persons dying every week is 78·4, what number will die in the year of 365 days, 5 hours, 48 minutes, 57 seconds?
16. If £96·54 represent the value of an acre of land on a certain estate, what is the worth of the whole, consisting of 1864 acres, 3·45 roods?
17. Add the rent of 53·7219 acres of land at £4·12 per acre per annum to that of 4·05 acres at £3·75 per annum.

DIVISION.

281. Whenever one decimal fraction has to be divided by another, if ciphers be added to that which has fewer places of decimals until both have the same number, the decimal points may be omitted, and the dividend and divisor will form the numerator and denominator of a vulgar fraction, which may be reduced to a decimal by the rule given in (262).

282. For by (228) whenever two fractions have a common denominator the one may be divided by the other, by simply dividing the numerator of the dividend by the numerator of the divisor. Therefore in dividing one decimal expression by another it is only necessary to bring both to the same name, and then the process of division will be the same as the division of integers. Thus, to divide 1·5 by ·003 is to find how many times ·003 or $\frac{3}{1000}$ is contained in 1·5 or $\frac{15}{10}$. But 1·5 or $\frac{15}{10} = \frac{1500}{1000}$. Therefore the sum may take this form, $\frac{1500}{1000} \div \frac{3}{1000}$, and this (228) is equivalent to $1500 \div 3$, or 500. Wherefore $1·5 \div ·003 = 500$.

By adding ciphers to 1·5 so as to make it a fraction of the same name as the other, the sum would at once have assumed a form for easy division, $1500 \div 3$.

Again, divide 27·9 by 16·08. Here, because $27·9 = \frac{279}{10}$, or $\frac{2790}{100}$, and $16·08 = \frac{1608}{100}$, the question is—Divide 2790 by 1608, *i.e.*, $\frac{2790}{1608}$. Here there is a vulgar fraction, which may be reduced to a decimal by (262), and the answer, though it may prove interminate, can be carried to any degree of accuracy required.

283. The following sums in Division of Decimals take the form of vulgar fractions, which may afterwards be resolved into decimals :—

$$70·960 \div 2·1 = \tfrac{70960}{1000} \div \tfrac{2100}{1000} = \tfrac{70960}{2100}.$$
$$·001 \div ·1 = \tfrac{1}{1000} \div \tfrac{100}{1000} = \tfrac{1}{100}.$$
$$2970·8 \div 1·0376 = \tfrac{29708000}{10000} \div \tfrac{10376}{10000} = \tfrac{29708000}{10376}.$$
$$2·0986 \div 1573·5 = \tfrac{20986}{10000} \div \tfrac{15735000}{10000} = \tfrac{20986}{15735}.$$
$$·029 \div 507·356 = \tfrac{29}{1000} \div \tfrac{507356}{1000} = \tfrac{29}{507356}.$$

Problems in Division of Decimals always assume one of the following forms :—

284. CASE I.—WHEN THE DIVISOR IS A WHOLE NUMBER—

RULE.

> Divide by the whole number, placing the first figure of the quotient in the same decimal place as the figure of the dividend from which it was obtained. Add ciphers, and continue the operation as far as may be necessary.

Example.—Divide 234·729 by 8, and 5·4726 by 12.

$$\begin{array}{r} 8)\underline{234{\cdot}729} \\ 29{\cdot}34112\ \ldots\ldots \end{array} \qquad\qquad \begin{array}{r} 12)\underline{5{\cdot}4726} \\ {\cdot}45605 \end{array}$$

In the former of these cases 23 is divided by 8 and a quotient 2 is obtained. But the 23 represents 23 tens, wherefore the eighth part of 23 represents a number of tens, and the 2 must stand in the tens place. In the second case, because the 54 are tenths, the question, what is the 12th part of 54, gives the answer in tenths. The same answers would have been obtained if we had reduced the divisor into a fraction having the same denominator as the dividend, but the process would in this case have been more tedious.

Thus, $234{\cdot}729 \div 8 = \frac{234729}{1000} \div \frac{8000}{1000} = \frac{234729}{8000} = 29{\cdot}34112\ .\ .$

And, $5{\cdot}4726 \div 12 = \frac{54726}{10000} \div \frac{120000}{10000} = \frac{54726}{120000} = {\cdot}45605.$

EXERCISE LXXXV.

1. 37·58 ÷ 6; 4·096 ÷ 17; 2158·9 ÷ 143.
2. 12·198 ÷ 4; 3·72096 ÷ 12; 41·793 ÷ 271.
3. 105·7096 ÷ 127; 3·01869 ÷ 500; 2108·962 ÷ 423.
4. 5079·638 ÷ 65; 2109·683 ÷ 49; ·0738 ÷ 3.
5. ·139655 ÷ 28; 4172·98 ÷ 37; 1·0076 ÷ 57.
6. ·031712 ÷ 227; 61·087 ÷ 35; 2·0198 ÷ 7.
7. Seven persons gained £374·56, what was each person's share?
8. A field of 15·42 acres, paying a rent of £18·425, was allotted among 16 labourers; what was the size of each man's allotment, and what had he to pay for it yearly?

DIVISION.

281. Whenever one decimal fraction has to be divided by another, if ciphers be added to that which has fewer places of decimals until both have the same number, the decimal points may be omitted, and the dividend and divisor will form the numerator and denominator of a vulgar fraction, which may be reduced to a decimal by the rule given in (262).

282. For by (228) whenever two fractions have a common denominator the one may be divided by the other, by simply dividing the numerator of the dividend by the numerator of the divisor. Therefore in dividing one decimal expression by another it is only necessary to bring both to the same name, and then the process of division will be the same as the division of integers. Thus, to divide 1·5 by ·003 is to find how many times ·003 or $\frac{3}{1000}$ is contained in 1·5 or $\frac{15}{10}$. But 1·5 or $\frac{15}{10} = \frac{1500}{1000}$. Therefore the sum may take this form, $\frac{1500}{1000} \div \frac{3}{1000}$, and this (228) is equivalent to $1500 \div 3$, or 500. Wherefore $1·5 \div ·003 = 500$.

By adding ciphers to 1·5 so as to make it a fraction of the same name as the other, the sum would at once have assumed a form for easy division, $1500 \div 3$.

Again, divide 27·9 by 16·08. Here, because $27·9 = \frac{279}{10}$, or $\frac{2790}{100}$, and $16·08 = \frac{1608}{100}$, the question is—Divide 2790 by 1608, *i.e.*, $\frac{2790}{1608}$. Here there is a vulgar fraction, which may be reduced to a decimal by (262), and the answer, though it may prove interminate, can be carried to any degree of accuracy required.

283. The following sums in Division of Decimals take the form of vulgar fractions, which may afterwards be resolved into decimals :—

$$70·960 \div 2·1 = \tfrac{70960}{1000} \div \tfrac{2100}{1000} = \tfrac{70960}{2100}.$$
$$·001 \div ·1 = \tfrac{1}{1000} \div \tfrac{100}{1000} = \tfrac{1}{100}.$$
$$2970·8 \div 1·0376 = \tfrac{29708000}{10000} \div \tfrac{10376}{10000} = \tfrac{29708000}{10376}.$$
$$2·0986 \div 1573·5 = \tfrac{20986}{10000} \div \tfrac{15735000}{10000} = \tfrac{20986}{15735}.$$
$$·029 \div 507·356 = \tfrac{29}{1000} \div \tfrac{507356}{1000} = \tfrac{29}{507356}.$$

Problems in Division of Decimals always assume one of the following forms :—

286. CASE III.—WHEN THE DIVIDEND HAS A GREATER NUMBER OF DECIMAL PLACES THAN THE DIVISOR—

RULE.

Divide as in whole numbers. Mark off in the answer as many decimal places as the dividend contains more than the divisor. Add ciphers, and carry the answer to any place of decimals required.

It would add to the trouble of calculation in this case if we were to bring both divisor and dividend to a common denominator. Thus if it be required to divide 7·9384 by ·8: this means (282), divide $\frac{79384}{10000}$ by $\frac{8000}{10000}$, or divide 79384 by 8000. If this were done the answer would be in whole numbers. But if instead of dividing by 8000 we divide by 8, and then when the answer is obtained we remember that it is 1000 times too much, and point off 3 figures accordingly, the sum will be worked more easily.

Example I.—Divide 27·3476 by ·15.

```
15)27·3476000
   ___________
   182·31733 . . . .
```

Because the dividend has 7 places of decimals and the divisor only 2, the answer has 5, or 7 − 2.

Example II.—Divide 34·79628 by 2·5.

```
25)34·79628(13·9185
   ·97
    229
     46
     212
      128
        3
```

Answer, 13·9185: because as dividend has 5 places of decimals and divisor 1, quotient must have 4.

287. *Observation.*—The reason of this rule will be further evident from the following considerations:—

I. Division is the inverse process of Multiplication. It has been shown in (279) that the product of any two decimal fractions must have as many decimal places as there are in both of the factors. But in Division the divisor is one factor and the quotient the other, their product being the dividend (129). Hence, as the number of decimal places in the dividend equals the sum of the number of

decimal places in divisor and quotient, it follows that the number of decimal places in the quotient is found by taking the difference between the number in the dividend and the number in the divisor.

II. It was shown in (225) that to divide one fraction by another is to take the reciprocal of the divisor, and proceed as in Multiplication.

Therefore to divide ·00162 by ·006, or to divide $\frac{162}{100000}$ by $\frac{6}{1000}$, is the same thing as to multiply $\frac{162}{100000}$ by $\frac{1000}{6}$.

But $\frac{162}{100000} \times \frac{1000}{6} = \frac{162}{6} \times \frac{1000}{100000}$.

And $\frac{162}{6} = 27$, and $\frac{1000}{100000} = \frac{1}{100}$.

Therefore $\cdot 00162 \div \cdot 006 = 27 \times \frac{1}{100} = \cdot 27$.

Because the dividend has 5 places of decimals its unexpressed denominator is 10^5, and because the divisor has 3 places of decimals the 6 is understood to be divided by 10^3. Now, because $\frac{10^5}{10^3} = 10^2$, the denominator of the quotient must be 100, or 10^2. Therefore the excess of the number of places in the dividend over that number in the divisor always shows how many places are to be in the quotient.

Exercise LXXXVII.

1. $\cdot 0001 \div \cdot 01$; $7\cdot 9285 \div \cdot 45$; $2\cdot 01 \div 1\cdot 7$.
2. $51\cdot 78 \div 1\cdot 1$; $3\cdot 0724 \div 178\cdot 36$; $40\cdot 735 \div 185\cdot 5$.
3. $720\cdot 397 \div 21\cdot 8$; $615\cdot 821 \div 2\cdot 4$; $3\cdot 271 \div \cdot 09$.
4. $1\cdot 2748 \div \cdot 53$; $61\cdot 423 \div \cdot 01$; $39\cdot 7286 \div 5\cdot 7$.
5. $11\cdot 723 \div \cdot 6$; $20\cdot 1783 \div 31\cdot 562$; $4\cdot 0198 \div 27\cdot 3$.
6. $14\cdot 05 \div 14\cdot 5$; $3\cdot 7298 \div 1\cdot 27$; $4\cdot 0198 \div \cdot 62$.
7. $(802\cdot 7 \times 4\cdot 6) \div 1\cdot 3$ of 7; $207\cdot 61 + 1\cdot 98 \div 7\cdot 15$.
8. $\dfrac{12\cdot 346 \times \cdot 017}{2\cdot 7 + 19\cdot 8}$; $\dfrac{62\cdot 704 + 3\cdot 001 - \cdot 987}{4\cdot 1235 - \cdot 68}$.
9. $\dfrac{21\cdot 307 \times 6\cdot 819}{2\cdot 3 \text{ of } 7\cdot 98}$; $\dfrac{(40\cdot 6 + 7\cdot 1) \times (3\cdot 029 - \cdot 1874)}{6\cdot 27 + 8\cdot 53 - 7\cdot 1}$.
10. Divide 3·14159 separately by 158, by 70·05, and by ·613.

Exercise LXXXVIII.

Work the last six sums in Exercise LXV. so as to obtain the answers in a decimal form.

** CONTRACTED METHODS OF MULTIPLICATION AND DIVISION.

288. In many problems in Multiplication or Division of Decimals the answer is only required to be accurate as far as a certain decimal place. In such cases some time would be wasted if we were to work out the whole sum according to the rules just given. For example, suppose it is required to find the product of 23·7286 and 414·793; it appears by (279) that the answer will extend to the 7th place of decimals, one of the factors having 3 and the other 4 places. But if for the purpose in hand it is only necessary to have an answer true to the 3rd place, it is evident that some part of the operation is useless and might be avoided.

Uncontracted process.

```
   414·793
    23·7286
   ---|----
   248|8758
  3318|344
  8295|86
290355|1
1244379|
829586 |
-------|----
9842·457|1798
```

Contracted process.

```
   414·793
    23·7286
   --------
    249
   3318
   8296
 290355
1244379
829586
--------
9842·457
```

In the first example the figures to the right of the vertical line are unnecessary, as the answer is not required to extend beyond the third place. In the second example no part of the multiplication has been performed which is not needed in the answer. Thus in the first line, to have multiplied the final figure 6 by that above it (3) would have produced an answer in the 7th place. But as the answer required is to be in the 3rd place, we multiply the 6 by that figure of the multiplicand which will give a product in the 3rd place. This figure is four places to the left of the 3, and we therefore begin the sum by taking 6 × 1. If we only did this, however, the answer would not be correct, for on looking at the two next figures of the multiplicand we see that the number 28 would come in the next place. But (271) because 30 is nearer 28 than 20 is, we carry 3 into the thousandths place. In the line below it we observe that the number of ten-thousandths was 63, we therefore only carry 6 into the thousandths place and neglect the 3. In the third line the number of ten thousandths is 18. We call this 20 rather than 10, and so carry 2 to the thousandths place. Proceeding in this way *we avoid* the trouble of setting down figures which do not contribute *to the answer.*

CONTRACTED RULE FOR DECIMAL MULTIPLICATION.

289. Begin with the right-hand figure of the multiplier and multiply by it that digit of the multiplicand which will give a product in the place required. Find what would have been carried from the place to the right of it, and add this to the first product.

Take the second figure of the multiplier and find the product of it and the digit of the multiplicand which stands to the right of that which was formerly chosen. Put the unit of this product into the required decimal place after accounting for the remainder from the right as before.

At each line take one more figure of the multiplicand into the product; proceed as before, and add up the results as in ordinary Multiplication.

EXERCISE LXXXIX.

(a). Find the following products to the third decimal place :—

1. 17·302 × ·579; 60·852 × 19·7; ·203 × 17·98.
2. 106·728 × 3·1957; 72·49 × 10·87632; 4·1985 × 2·1743.
3. 70·96 × 8·074; 32·75 × 41·7209; 62·8145 × 3·172.
4. 7·0968 × 3·12; 7·145 × 6·1437; 8·179 × ·2146.
5. 7·986 × 2·09 × 4·723; 6·798 × 4·072 × ·5.
6. 5·189 × 64·3274; 8·274 × 35·2968.
7. 4·1725 × 31·72; 4·096 × ·38 × 1·792.

(b). Find the products of the first six sums in Exercise LXXXIV., contracting each answer by two digits.

290. DIVISION.—When the quotient of a Division sum is only required to be true to a certain decimal place, a similar method may be adopted.

Example.—Divide 7·9362 by 2·7451 to four places of decimals.

Uncontracted method.

```
2·7451) 7·9362 (2·8910
        54902
        24460|0
        21960|8
         2499|20
         2470|59
           28|610
           27|451
            1|1590
```

Contracted method.

```
2·7451) 7·9362 (2·8910
        54902
        24460
        21960
         2499
         2470
           28
           27
            1
```

6. How much is 14·973 shillings of a ten pound note? an much of a farthing?
7. Reduce 3·35 shillings to the decimal of £1, of 10s., and
8. Express ·5 shillings + ·7 crowns + £·15 as a decimal of a
9. What fraction of 11 cwt. is ·297 lb.?
10. How much of a bushel is ·397 of a pint?
11. Reduce 1·27 oz. to decimals of a grain, a scruple, and
12. How much of a square mile is ·39 of a rood?

294. CASE III.—TO FIND THE EQUIVALENT VALUE TO A DECIMAL IN ORDINARY CONCRETE NUMBERS—

RULE.

Reduce only the fractional part of the expression Case I. reserving the whole number as part answer. Continue to reduce the fractions unt lowest denomination is reached. The ser whole numbers will form the answer.

Example I.—What is the actual value of 23·568 of 1s.?
23·568 of a shilling = 23 shillings + ·568 of 1 shilling.

But by (292) ·568 of 1s. = ·568 × 12 = 6·816 of 1 penn pence + ·816 of a penny.

Again ·816 of a penny = ·816 × 4 = 3·264 of a farthin farthings + ·264.

Hence 23·568s. = 23s.; 6·816d. = 23s. 6d.; 3·264 farthi £1 3s. 6¾d. + ·264 farthings.

Example II.—Reduce 29·3814 cwt. to the ordinary form.

29·3814	cwt.
4	
1·5256	qrs.
28	
42048	
10512	
14·7168	lbs.
16	
11·4688	ounces

Answer, 29 cwt. 1 qr. 14 lbs. 11·468 oz.

Example III.—Reduce a mile to the ordinary forr

·178	of a mile
8	
1·424	furlongs
40	
16·960	poles
5½	
5·280	yards
3	
·840	feet
12	
10·080	inches

Answer, 1 furlong, 16 p feet, 10·08 inches.

EXERCISE XCII.

Reduce the following to the ordinary forms of expression:—

1. 17·914 weeks; ·178 gallons; 32·746 weeks.
2. 2·074 guineas; £5·728; 16·474 shillings.
3. £17·5028; £1·627; £4·78.
4. £·68; £2·47; £1·28
5. £103·4792; £2·18634; £7·096.
6. £7·8425; £303·796; £2·0872.
7. 29·324 cwt.; 5·87 cwt.; 2·97 quarters.
8. ·07 of £2 10s.; £·972916; £·10875.
9. 3·27 days; 4·728 weeks; 39·278 hours.
10. 1·29 miles; 27·28 furlongs; 3·05 leagues.
11. 27·38 yards; 4·196 poles; 282·72 miles.
12. 37·25 acres; 48·972 acres; 25·34 sq. roods.
13. ·13·38 sq. yards; 14·23571 sq. miles: ·978 acres.
14. 17·84 gallons; 212·305 gallons; 4·07 quarts.
15. 29·732 quarters of corn; ·08 quarters; 53·197 quarters.
16. 18·7 English ells; 3·207 French ells; 132·72 nails.

295. CASE IV.—WHEN A CONCRETE NUMBER IS GIVEN IN THE ORDINARY FORM, TO EXPRESS IT DECIMALLY—

RULE.

Begin with the quantity of the lowest denomination named. Treat this separately, and divide it by as many of the less as make one of the next higher name. Add the whole number of this higher name to the fraction thus found, and divide again so as to reduce this to the denomination next above. Continue to divide and to add in the whole numbers one by one, until the required denomination is reached.

Example I.—Express 2 lbs. 7 oz. 3 dwt. 15 grains troy as decimals of 1 lb.

Here (233) 15 grains = $\frac{15}{24}$ = ·625 of 1 dwt. ∴ 3 dwt. 15 grains = 3·625 dwt.

But 3·625 dwt. = $\frac{3·625}{20}$ = ·18125 of 1 oz. ∴ 7 oz. 3 dwt. 15 grains = 7·18125 oz.

Again 7·18125 oz. = $\frac{7·18125}{12}$ = ·59927083 of 1 lb. ∴ 2 lb. 7 oz. 3 dwt. 15 *grains* = 2·59927083 lbs.

The same sum would be more conveniently worked in this form—

2 lbs. 7 oz. 3 dwt. 15 grains.

24) 15	grains
20) 3·625	dwt.
12) 7·18125	oz.
2·59927083	lbs.

Example II.—Reduce £17 9s. 8¾d. to the decimal of 1s.

4) 3	farthings	
12) 8·75	pence	= 8¾d.
9·72916	shillings	= 9s. 8d¾.
340	shillings	= £17
349·72916	shillings	= £17 9s. 8¾d.

Example III.—Reduce 7 qrs. 5 bush. 3 pecks 1 gal. 2 quarts to the decimal of a quarter.

4) 2	quarts	
2) 1·5	gallons	= 1 gal. 2 qts.
4) 3·75	pecks	= 3 pecks 1 gal. 2 qts.
8) 5·9375	bushels	= 5 bush. 3 pecks 1 gal. 2 qts.
7·7421875	quarters	= 7 qrs. 5 bush. 3 pecks 1 gal. 2 qts.

Exercise XCIII.

1. Reduce £17 10s. 9¾d., £2 5s. 7½d., and £8 15s. 4¼d. to the decimal of £1.
2. Reduce £25 18s. 7½d., £1 10s. 5¾d., and £297 10s. 7½d. to the decimal of £1.
3. Reduce 17 cwt. 3 qrs. 2 lbs., 3 qrs. 9 lbs., and 427½ lbs. to the decimal of 1 cwt.
4. Reduce 294 cwt. 3 qrs. 17 lbs., 10 cwt. 27 lbs., and 17 cwt. 9 lbs. to the decimal of 1 qr.
5. Reduce 17 lbs. 6 oz., 2 qrs. 1 lb., and 3 cwt. 1 qr. to the decimal of 1 lb.
6. Reduce 13 oz. 4 drs., 29$\frac{3}{7}$ lbs., and 4 cwt. 2 qrs. 7 lbs. 4 oz. to the decimal of 1 lb.
7. Reduce 5 days 12 hours 25 mins. 37 secs. to the decimal of a week.
8. Reduce 3 weeks 3 days 23 hours to the decimal of a week.
9. Reduce 12 hours 55 mins. 23⅓ secs., and 17 hours 47 mins. 33 secs. to the decimal of a day.
10. Reduce 7 oz. 4 dwts. to the decimal of a pound troy, and 9 oz. 2$\frac{3}{7}$ drs. to the decimal of a pound avoirdupois.
11. Reduce 8s. 7¼d. to the decimal of a guinea, and £3 17s. 6d. to that of a shilling.
12. Reduce 19 qrs. 2 bush. 2 pecks to the decimal of a quarter, and 7 quarts 3 pints to that of a gallon.

SECTION XI.—DECIMAL MONEY.

296. The Arithmetic of concrete quantities is more difficult than that of abstract numbers, because the units are not subdivided decimally, but follow in each case some special rule. Thus the principal units of weight, of time, of linear measure, of value, are variously subdivided thus :—

1 cwt. = 4 qrs. = 112 lbs. = 1792 ounces.

1 week = 7 days = 168 hours = 10080 minutes.

1 mile = 8 furlongs = 320 poles = 1760 yards = 5280 feet = 63360 inches.

£1 = 20 shillings = 240 pence = 960 farthings.

In the chapter on Weights and Measures (Appendix) some reasons will be found for these irregular divisions; but it is evident that if we had been accustomed to divide each of these principal units into tenths, hundredths, thousandths, &c., the addition, subtraction, multiplication, and division of concrete numbers would be much easier than it now is. If for example we had a name for the tenth and hundredth of a ton, instead of the twentieth and the eightieth, we should be spared the trouble of asking ourselves a question in reduction at every step of a sum in avoirdupois weight, and the whole process would exactly resemble Simple Arithmetic. But long established usage and habits have made it very difficult to alter the familiar weights and measures which are current among us, and such a complete decimal system as has been adopted in France, and is described in the Appendix, has scarcely been proposed in this country, and will certainly not come into general use for many years.

297. The first change will probably affect our coinage, for this could be decimally adjusted with less difficulty than any other concrete quantity. The 10th of a pound is 2 shillings; and although we have no exact equivalent in our present coinage for the hundredth of a pound, yet the farthing, which is $\frac{1}{960}$ of a pound, differs very little from $\frac{1}{1000}$. If therefore we retain the pound as the unit, it becomes necessary to give a name to the value now called 2 shillings, to have a new coin and a new name for the tenth of 2 shillings, or the hundredth of a pound, and to depreciate the value of the farthing 4 per cent. so that it shall become $\frac{1}{1000}$ of the pound. If, however, we retain the farthing as the unit, we shall require a new name for 10 farthings (or $2\frac{1}{2}$d.), another name and a new coin for 100 farthings

(2s. 1d.), while 1000 of the unit would be equivalent to £1 0s. 10d. of our present money. By the first system we should retain the pound as the unit, and all other coins would be measures or aliquot parts of it; by the second we should retain the farthing as the unit, and all other coins would be multiples of it. This latter would have one advantage, for all the sums represented by the new names could be paid by our present money, while the hundredth and thousandth of a pound cannot be represented in value at all without entirely new coinage. Nevertheless, the first system would retain the pound and shilling at their present values, while this would be impossible by the second. Since, therefore, all our most important accounts are kept in pounds, it is thought less inconvenient to adopt a system which will cause the penny, the farthing, and the fourpenny piece to disappear, but will leave the pound and the shilling unaffected, than to take one which will supersede the pound as the unit, though retaining the smaller coins. Accordingly it has been proposed by a Committee of the House of Commons, that whereas at present

£1 = 20 shillings = 240 pence = 960 farthings,

an arrangement shall be made whereby the pound shall be thus divided:—

£1 = 10 tenths = 100 hundredths = 1000 thousandths.

298. It is not yet determined what names shall be given to these aliquot parts of the pound. Names indicating the relation of the several values to each other would be the best; and it has been suggested that plain Saxon contractions of the words tenths, hundredths, and thousandths, such as *tens*, *huns*, and *thous*, would serve the purpose. The first coin, however, the tenth of a pound, is already in use, and has been unwisely denominated a florin,* and the public taste seems rather in favour of Latin words, such as the French have adopted (*decime*, *centime*, *milliéme*). Our system will probably be

£1 = 10 florins = 100 cents = 1000 mils.

Such an expression as £17·387 would, on this plan, mean £17, 3 florins, 8 cents, 7 mils, and might be multiplied and divided as

* We think this name unfortunate for two reasons:—I. Because it carries with it *no allusion to* the decimal system; and II. Because in so far as it suggests any *meaning at all it is* misleading, for none of the coins bearing that name and in use [on] *the Continent of* Europe have the same value.

readily as the simple number 17387; because ten of any one place would equal one in the place to the left.

299. As the adoption of such a coinage, though very desirable, may yet be delayed for some time; it is important to know the readiest method of applying the decimal system to the keeping of accounts and of computing the money current among us at present by the rules of decimals. Taking £1 as the unit—

s.	*d.*	
10	0	= ·5
5	0	= ·25
2	6	= ·125
2	0	= ·1
1	0	= ·05
0	6	= ·025
0	1	= $\cdot 0041\dot{6}$
0	0¼	= $\cdot 001041\dot{6}$

Of these several values some are much more easily expressed as decimals of £1 than others. All except the two last-named might be retained in a decimal system. The relation between these two coins and £1 cannot be expressed by the help of 10 or any power of 10. Neither a penny nor a farthing can therefore be retained as units of value on the proposed plan. But as the farthing is now $\frac{1}{960}$ of £1, we may represent it as ·001 of £1 without serious error, except in cases where the decimal expression has to be multiplied considerably.

300. When accuracy is only required to the third place of decimals it is sufficient to remember this short table.

s.	*d.*	
2	0	= ·1
1	0	= ·05
0	6	= ·025
0	0¼	= ·001

For every 2 shillings in the sum we have ·1; for every shilling, ·05; for half a shilling, ·025; and for a farthing, ·001. Hence a sum of money, £21 18s. 7¾d. may be written as 21 + ·9 + ·025 + ·007 = 21·932. We take 9 in the first place to represent 18s., ·25 in the second and third places for the 6d., and 7 for the additional farthings.

Suppose, now, it were required to divide this sum by 6, the same would be done very readily thus—

6)21·932

3·655 =£3, 6 florins, 5 mils, 5 cents = £3 13s. 1¼d.—

the decimal expression being transformed into £3, 12s., 1s., 5 farthings, or £3 13s. 1¼d. But if it had been required to multiply this sum of money by 6, it is evident that whatever error is made in calling 7 farthings ·007 of £1 is also multiplied by 6. In adopting the following rules it must, therefore, be remembered that the former, which secures accuracy to the third place, will suffice in all ordinary cases in which short Addition sums, or almost any sums in Subtraction and Division, are effected; but that the latter will be needed whenever the sum of money to be dealt with requires to be multiplied.

301. Case I.—To reduce any sum of money to a decimal form which shall be true to the third place—

RULE.

Take the pounds as whole numbers and fill up the three places of decimals as follows:—100 for every 2 shillings, 50 for a shilling, 25 for 6 pence, and 1 for every additional farthing.

Example I.—£153 15s. 4¼d. = 153·767. Here 14s. = ·700, 1s. = ·050, and 17 farthings = ·017, and 700 + 50 + 17 = 767.

Example II.—£27 5s. 6¾d. = £27·278. Because 4s. = $\frac{200}{1000}$, 1s. = $\frac{50}{1000}$, 6d. = $\frac{25}{1000}$, and ¾d. = $\frac{3}{1000}$.

Exercise XCIV.

Reduce the following sums to decimals true to the third place:—

1.	£2 14s. 8d.;	£3 10s. 9½d.;	£27 4s. 6¾d.
2.	£7 18s. 4½d.;	£235 15s. 3d.;	£228 6s. 7½d.
3.	£14 17s. 9d.;	£628 4s. 7d.;	£5 13s. 4¼d.
4.	£1 11s. 7½d.;	£274 16s. 3d.;	£2 0s. 9¾d.
5.	£2 7s. 4¼d.;	£8 11s. 2¾d.;	£17 6s. 9½d.
6.	£8 2s. 4½d.;	£37 2s. 3¼d.;	£11 9s. 4¼d.

Exercise XCV.

Reduce the following decimals to ordinary numbers by this rule:—

1. £19 8 florins 6 cents 4 mils; £25 3 florins 7 cents; £176 1 florin 3 cents 5 mils.

2.	£17·528;	£1·064;	£18·371;	£20·965;	£8·27.
3.	£16·721;	£4·128;	£3·074;	£4·106;	£2·093.
4.	£4·696;	£18·325;	£71·421;	£58·372;	£41·629.
5.	7·284;	£5·901;	£8·623;	£5834·721;	£209·645.
6.	£·872;	£3·96;	£·047;	£814·756;	£203·871.
7.	£·97;	£·083;	£·46;	£8·695;	£23·72;	£·072.

Exercise XCVI.

Work the first four sums in Exercise XII.; the first six in Exercise XXIX.; and the first three in Exercise XXX., by this method.

302. CASE II.—WHEN IT IS REQUIRED TO MAKE THE EXPRESSION ACCURATE TO ANY GIVEN PLACE OF DECIMALS—

RULE.

Express the sum of money as far as sixpence by the rule just given; but for every penny above an even sixpence place $41\dot{6}$ in the third and following places, and for every odd farthing place $1041\dot{6}$ in the third and following places, as far as required.

303. *Observation.*—The table given in (299) shows that the former rule gives perfectly accurate results as far as 6d., but that the value of odd pence and farthings cannot be expressed without descending lower in the decimal scale. As in both cases repeating decimals occur it is impossible to be entirely accurate even then, but it is easy to carry the calculation so low that the error shall be as slight as we please.

Examples.—Reduce £17 13s. 8¼d. to decimals true to the 5th place, and £270 3s. 2¾d. true to the 7th place.

	£	*s.*	*d.*		£
I.	17	0	0	=	17
	0	12	0	=	·6
	0	1	0	=	·05
	0	0	6	=	·025
	0	0	2	=	·00832
	0	0	0¼	=	·00104
					17·68436

	£	*s.*	*d.*		£
II.	270	0	0	=	270
		2	0	=	·1
		1	0	=	·05
	11 × $·001041\acute{6}$				·0114582
					270·1614582*

EXERCISE XCVII.

Reduce the following to decimals true to the 6th place:—

1. £17 10s. 3½d.; £27 13s. 4¾d.; £109 8s. 6½d.
2. £273 5s. 1¼d.; £12 6s. 8¾d.; £1 6s. 3¼d.
3. £28 9s. 4¼d.; £1 0s. 7¼d.; £23 5s. 6¾d.
4. £17 0s. 8¼d.; £29 16s. 3¼d.; £18 0s. 9d.

EXERCISE XCVIII.

Work the first four sums in Exercise VIII., and the first five in Exercise XX., by this method.

* This process, though apparently long, is much simpler when performed mentally than when laid out at length on paper. A very little practice will enable a student to use both rules with great facility.

Questions on Decimal Fractions.

What is meant by the expression Decimal Fraction? In what respect does this Fractional Notation resemble the common Notation of Integers? What is the main difference between the two? What purpose is served by the number 10 in Integral Notation? What in Fractional Notation?

In the following line of figures what is the separate value of each, 7928·534728? Compare the values of two figures, one of which is 5 places to the right and the other 5 places to the left of the decimal point. Describe the use of the cipher (0) in Fractional Notation. Why are decimal fractions very readily multiplied or divided by 10 or any power of 10? Give the rule which applies to such cases.

What is the great advantage possessed by decimal over vulgar fractions? Has a decimal fraction any denominator? if so, how do you determine what it is? What process in Decimals is analogous to the reduction to a common denominator of Vulgar Fractions? Describe it, and state the rule. How are decimals converted to vulgar fractions and vulgar to decimals? Give the reason for the rule in each case. Show in what way this rule resembles that of ordinary Division or Reduction.

What are Recurring Decimals? How many kinds of them are there? In what cases do they occur? How may they be reduced to vulgar fractions? Demonstrate the reason of the rules. What kind of vulgar fractions always give recurring decimals?

How do you determine the position of the point in the answer to a sum in Addition or Subtraction of Decimals? Why? In what way do the Addition and Subtraction of Decimals differ from, and in what way do they resemble, the Addition and Subtraction of Integers?

Where should the decimal point be placed in a product or in a quotient? Why? Show how the Rule for Multiplying Vulgar Fractions applies to Decimals. What is there in Decimals corresponding to the multiplication of the denominator in Vulgar Fractions?

When is Division of Decimals the same as in whole numbers? What general rule would apply equally to all possible cases of Division of Decimals? Why is not the same rule applied in all cases? State the different special rules of Division and give a reason for the use of each. How is the principle of reciprocals illustrated in Decimal Division? Give an example.

Why is it better to contract certain sums in Multiplication and Division? State the processes, and show when each should be used. What sort of error, if any, is liable to occur in these processes?

State the axiom assumed in the rule for reducing a certain value into the decimal of a greater or a less. State the rule in each case, and give an example showing how the axiom applies. If a certain fraction of a Troy pound had to be reduced to the decimal of an Avoirdupois pound, what rule would be necessary?

Compare the advantages of a decimal system having £1 for the unit with those of one which has a farthing for the unit. What is the readiest way of converting our present money into decimals of £1? What sort of error will occur in using the rule, and when?

MISCELLANEOUS EXERCISES ON FRACTIONS.

1. What is the value of £54·732, and of £763·824?

2. Add together $3\frac{1}{5}$, $7\frac{1}{3}$, 25·687, and 1·666.

3. A can do $\cdot\dot{6}$ of a piece of work in 4 hours, B can do ·75 of the remainder in 1 hour, and C can then finish it in $\frac{1}{3}$ of an hour; in what time can A, B, and C do it?

4. Add together $\frac{1}{3}$ of $\frac{5}{7}$; $\frac{1}{4}$ of $\frac{19}{21}$; and $\frac{5}{6} \div 1\frac{5}{9}$.

5. Reduce 3s. $4\frac{1}{2}$d. to the fraction and to the decimal of £1.

6. Reduce ·4786 days to hours, minutes, and seconds, and find the value of 173·25 yards at 5·25s. per yard.

7. Reduce $\frac{2}{3}$ of a day to the decimal of a week and of a year.

8. Simplify each of the following expressions:—

$$\frac{9\frac{5}{7}}{13{\cdot}25}; \quad \frac{\frac{2}{3}\text{ of }\frac{4}{5}}{\cdot 15 + \frac{1}{8}}; \quad \frac{4\frac{4}{7}}{2{\cdot}3}; \quad \frac{\cdot 284}{\frac{2}{7} \times \frac{1}{3}}; \quad \frac{6\frac{5}{7}}{2{\cdot}75}.$$

9. What is the difference between $\frac{2}{3}$ of $\frac{4}{7}$ of a pound, and $\frac{6}{7}$ of a shilling?

10. Add together $\frac{4}{7}$ of a pound, $\frac{2}{9}$ of a shilling, and $\frac{5}{12}$ of a penny.

11. Add $\frac{2}{3}$ of a pound, $\frac{3}{14}$ of a guinea, and $\frac{2}{5}$ of 4s. 7d.

12. Multiply and divide 60 by ·00048, also 7·29 by ·0028.

13. If to one person a testator bequeaths $\frac{9}{20}$ of his property, to another $\frac{1}{4}$, and to another £300, what is the value of his property?

14. Add together the circulating decimals 0·53434, &c., and ·0465858, &c., and subtract the sum from $1\frac{1}{3}$.

15. Subtract $\frac{3}{7}$ from 1·1 and divide the remainder by 0·1.

16. What is the value of 72 lbs. 3 oz. at £4 3s. 2d. per cwt., and of 6 cwt. 1 qr. 19 lbs. at £7 10s. per cwt.?

17. Find a series converging to $\frac{251}{764}$, and also to 3·14159.

18. What is the rent of 311 acres 2 roods 26 poles at £1 2s. 9d. per acre?

19. If a vessel is two-thirds full, and after 70 pints are drawn off is found to be three-eighths full, how much could it contain?

20. If ·7 of anything cost £29 8s. 7d., what share will be worth £15 10s. 4d.?

21. Find the greatest common measure of 236·511 and 37·429, also of 20·75 and 11·39.

22. The circumference of a circle is 3·14159 of the diameter, find the circumference of circles whose diameters are 13·7 feet, 1·96 yards, and 28·342 miles respectively.

23. What are the diameters of circles whose respective circumferences are 12·56 inches, 3·297 feet, and 11·08 miles?

24. Find a quarter's rent on 114·76 acres at £3·74 per annum per acre, and two years' rent on £47·5 acres at the same rate.

25. Work the following sums decimally:—

(a). 1172 at £1 8s. 9d.; 3274 at £7 6s. 6d.; 2093 at 16s. $7\frac{1}{4}$d.

(b). 2093·5 at £7 12s. 4d.; 812·53 at 17s. 6d.; 412·9 at £1 2s. 1d.

(c). 81·7 at £4 10s. $3\frac{1}{4}$d.; 519·6 at £17·28; 327·4 at £·68.

(d). 41·5 at £2·78; 8062 at £12·093; 712·34 at £·06.

26. If a man spends $\frac{5}{7}$ of his income in board and lodging, and ·125 in dress and amusements, and then can save £117 per annum, what is his income?

27. If a bath be supplied with water by two pipes, one of which alone would fill it in 20 and the other in 15 minutes, and if a discharging pipe would empty it in 25 minutes, how long will it be in filling if all three pipes are employed together?

28. If 7 persons take equal portions of a piece of cloth 19 yards long, express the share of each in decimals of an English ell.

29. If, in an election, 1,253 voted, and one candidate had a majority of 89 over the other, what fraction of the whole voted for each?

30. A French metre is 39·371 English inches. Express the following lengths in metres—3 miles 7 furlongs 12 poles, 5 miles 1250 yards, 17 miles 3 furlongs 2 yards 15 inches, and 7·2854 furlongs.

31. Multiply the square of $1\frac{5}{70}$ by $3\frac{15}{47}$, and give the answer in decimals.

32. If two persons do work in 10·5 days which one of them alone would finish in 17·75 days, in what time would the other do it?

33. A gallon contains 277·274 cubic inches, find the cubic contents of the following measures,—a gill, a pint and a half, a measure containing 3 quarts, and an 11 gallon cask.

34. When the rate of exchange is such that an English sovereign is worth 25·19 French francs, how would the following sums be expressed in French money:—£3 17s. $10\frac{1}{2}$d., £18 6s. 8d., and *£214 7s. $4\frac{1}{4}$d.?*

35. At the same rate what would be the English equivalents to 17·96 francs, 2183·45 francs, and 1069·4 francs?

36. If an ounce of gold be worth £$4{\cdot}1895\dot{3}$, what is the value of $\cdot 37\dot{5}\dot{3}$ lbs.?

37. Compare the number of revolutions made by two wheels, one of which is 3 feet, and the other 5 feet, in diameter, in rolling over 17 miles.

38. $\frac{3}{7}$ of an estate is sown with wheat, ·15 of it is meadow land, and the rest consists of 24 acres 2 roods 17 poles; find the whole area of the land.

39. If A owns ·48 of a ship and B the rest, and the difference in the value of their shares is £23·76, what is the worth of the ship?

40. If $\frac{5}{7}$ of an estate contains 250 acres, and is worth £1,003 17s. 1d., what is the extent and value of ·15 of it?

41. A Spanish dollar is worth $3{\cdot}\dot{3}$ shillings. Express $515\frac{3}{4}$ Spanish dollars in English money, and £728 13s. 4d. in Spanish.

42. The specific gravities of gold, silver, and copper, are respectively 19·36, 10·47, and 8·95. Find the weights of a lump of each, of the same bulk as a vessel of water weighing 7 lbs. 6 oz.

43. The year consists of 365·24224 days. In what time will the error of calling it $365\frac{1}{4}$ days amount to an error of one day?

44. The area of a circle is found by multiplying the square of the diameter by one fourth of 3·14159. Find the areas of circles whose diameters are respectively 12·16 feet, 73 inches, and 2·75 yards.

45. In the event of a change of coinage, the rate of postage would be necessarily altered from one penny either to 5 mils or to 4 mils. Compare the number of letters which could be sent for £47 11s. 6d. at the three several rates of 5 mils, one penny, and 4 mils.

46. £25 3s. 6d. are now received on an average per week, at a turnpike where $1\frac{1}{2}$d. is charged for every vehicle. What would the toll-contractor lose per week if the toll were lowered to 6 mils, and what would be the difference in his receipts if it were altered to 7 mils?

47. In a British school 70 boys on an average pay 2d. per week throughout the year, and 55 boys pay 3d.; what are the annual receipts, and what would they be if the former paid one cent and the latter one cent and a half instead?

RATIO AND PROPORTION.

SECTION I.—THEORY OF THE PROPORTION OF ABSTRACT NUMBERS.*

304. From (3) it appears that we possess no ideas of *absolute magnitude* of any kind.† All the notions we have of size, or weight, or of duration, or of any magnitude whatever, are *relative*. To form a clear notion of any such magnitude we must compare it with some one thing of the same kind which we have either arbitrarily chosen or which exists in nature. When we have made this comparison the result is what we call a *number*.

305. All Arithmetic is employed in establishing comparisons between some quantity supposed to be previously known, and some other quantity which is intended to be expressed. Thus, Integral Arithmetic takes unity as the standard of comparison, and considers all other magnitudes as multiples of it. Fractional Arithmetic takes unity as the standard of comparison, and considers it as separated or divided into parts.

But if instead of merely considering the relationship between a certain magnitude *and unity*, we compare any two numbers *with one another*, then the result of such comparison is called the *ratio*‡ of the two numbers.

* The student who has not read Section IX., on Multiplication and Division should do so very carefully before commencing this chapter.

† The fact that all our notions of magnitude are relative and not absolute will be more clearly seen from the use of the current words, great, small, many, few, tall short, &c. Not one of these words has any definite and invariable meaning. By a great house we mean a house greater than average houses. A small castle is one which is small compared with other castles. Yet the former, though properly called great may be absolutely less in size than the latter. The same number of people which might be called many in a garden would be few in a city, &c. In all our thoughts of magnitude we silently refer to some one familiar standard, and measure by comparison with it.

‡ *I.—The word* which Euclid employs here is λόγος, which has several meanings

306. There are two ways in which magnitudes of the same kind may be compared with each other.

I. When we ascertain by how much one magnitude exceeds the other. This is to be done by subtracting the less from the greater.

II. When we ascertain how many times one contains the other. This is to be done by dividing the one by the other.

$$\text{Thus, } 36 - 4 = 32, \qquad \text{and } 36 \div 4 = 9.$$

The result of the comparison of these two numbers by the former method is 32; the result of the comparison of the two numbers by the latter method is 9.

307. *Observation.*—Of these two methods of comparison the second is that which naturally suggests itself to the mind and which we habitually employ. If of two persons one possesses £9998 and the other £10000, we should think the difference very trifling, and should look upon them as equally rich; but if one possessed £3 and the other £5, we should not look upon the one sum as near to the other. Yet there is the same absolute difference here as in the former case.

308. The two kinds of relationship here described are very different, but the same word, Ratio, has been sometimes applied to both. The result of comparison by Subtraction is sometimes called the *arithmetical ratio* of two numbers, and the result of comparison by Division has been called the *geometrical ratio.* These terms, however, are very inappropriate, and will not be again employed here. The word *ratio* when standing alone is generally intended to represent the latter relationship only, and it is in this sense that it will be hereafter used. The comparison of two magnitudes by Subtraction, or Arithmetical Proportion as it is sometimes called, is discussed in the Section on Arithmetical Progression.

309. Ratio * is the relationship which subsists between

in Greek, but in its application to Mathematics is always used in the limited sense indicated by the definition in the text. The Roman word, "ratio," exactly represented its meaning. In English we have no term of the same signification, and therefore we employ this word, "ratio," for the purpose. The French use sometimes the word "raison," as equivalent to "ratio," but more frequently the general word "rapport," which is applied to both the kinds of relationship we have described.

* II. Euclid's definition, as commonly translated, is apparently more simple than this. "Ratio is the relation between two magnitudes of like kinds with respect to quantity." But it is evident that to ask what is the relation of two magnitudes

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two quantities of like kinds, with respect to the number of times one contains the other.

310. Whenever two numbers are compared in this way the first is called the Antecedent, and the second the Consequent of the ratio.

We express this relation by placing (:) between the two. Thus $7 : 12 =$ the ratio of 7 to $12 =$ the part which 7 is of 12. Here 7 is the antecedent and 12 the consequent.

311. *Observation I.*—Ratio can only subsist between magnitudes *of like kinds*. The reason which was given (31) why numbers cannot be added or subtracted unless they refer to the same things applies equally here. No two magnitudes can be compared which are not of the same sort. If we were asked to discuss the relation between 7 cwt. and 3 hours, the mind would refuse to entertain the question, and reject it as an absurdity.

312. *Observation II.*—Ratio has just been called the result of the comparison of two quantities or numbers. Before we can make such a comparison we must have a clear notion of the magnitude of each, but when we have made it we obtain a notion of a third magnitude, and this is what is called the "ratio" of the two others. It is quite distinct from the two former magnitudes. If I see one pole 3 yards high, and another 5 yards high, and I compare their heights together, I say that the one is to the other as 3 is to 5, or that the one measures $\frac{3}{5}$ of the height of the other. This notion, which I express as $(3 : 5)$ or $(\frac{3}{5})$, is not a notion of the height or length of either. It is quite distinct also from the number 3 or the number 5 considered separately. It is simply the result of comparing them together, and

with respect to quantity may be to ask *how much* greater one is than the other; and the answer to this question does not express the ratio of the two numbers in the ordinary sense. The word quantity seems better fitted to describe the former of the two relationships referred to in (306), and certainly does not represent Euclid's meaning. For the Greek word, πηλικότης, which he employs, is derived from πηλικος, which means, "how many times?" We have no one English word to express this, but in Latin the word "quot" was its exact equivalent. If from this word we had an English derivative "quotity," corresponding in its form to "quantity," it would express Euclid's meaning, which was—not "how much," but "how many times."

πόσος = quantus = how much; ποσότης = quantity = how much-ness.
πηλικος = quot = how many times; πηλικότης = quotity = how many times-ness.
It is this last-mentioned Greek word which Euclid uses, and we have expressed its meaning in the text by a somewhat circuitous phrase, because the word quantity conveys an erroneous impression.

represents the part which one is of the other, or the relation one bears to the other. Yet this relation is evidently a third magnitude, different from the two former. We should have had precisely the same result before our minds if we had compared the possessions of two persons, one of whom had £60 and the other £100. For £60 is the same part of £100 that 3 yards is of 5 yards; and as the expression 60 : 100, or $\frac{60}{100}$, represents the part which one sum of money is of the other, and the expression 3 : 5, or $\frac{3}{5}$, the part which one height is of the other, the two values 3 : 5 and 60 : 100 are equal, and the idea which they represent is precisely the same, although one is derived from the comparison of two sums of money, and the other from the comparison of two lengths.

313. Three important inferences may be deduced from these considerations:—

I. A ratio is never a concrete number, but is simply an abstract numerical notion, derived from the comparison of two magnitudes, and representing their mutual relations.

II. The two terms of a ratio are related to one another exactly as the two terms of a fraction; thus the expression $\frac{a}{b}$ means just the same as $a : b$, that is, the number of times that a contains b (182).

III. Every truth which may be affirmed of the numerator and denominator of a fraction applies equally to the antecedent and consequent of a ratio.

314. Hence all the propositions in (189 *et seq.*) will be found true if the words antecedent, consequent, and ratio be substituted for numerator, denominator, and fraction. *e.g.*,

If the antecedent of a ratio be increased or diminished, the ratio is increased or diminished in like manner; but if the consequent be increased the ratio is diminished, and if the consequent be diminished the ratio is increased.

Demonstrative Example.—6 : 2. Here the ratio is 3, but if we increase the antecedent 4 times it becomes 24 : 2, or 12.

General Formula.—The ratio $a : b = \frac{1}{m}$ of the ratio $ma : b$; and m times the ratio $= \frac{a}{m} : b$.

If both antecedent and consequent be increased or diminished the same number of times the ratio remains unaltered.

Demonstrative Example.—15 : 6. Here the ratio is $\frac{15}{6}$ or $2\frac{1}{2}$; if we multiply antecedent and consequent by 4, the ratio becomes $\frac{60}{24}$, or $2\frac{1}{2}$, and if we divide both by 3 it becomes $\frac{5}{2}$ or $2\frac{1}{2}$.

General Formula.—$a : b = ma : mb = a \div m : b \div m$.

315. Proportion is the equality of ratios.

If there be any four magnitudes such that the first is as many times greater or less than the second as the third is greater or less than the fourth, these four magnitudes are said to be in proportion.

Example.—Because 5 is the same part of 15 that 20 is of 60, the four numbers, 5, 15, 20, 60, are in proportion; that is, the ratio of 5 to 15 (5 : 15) is the same as the ratio of 20 to 60. This may be thus expressed: $\frac{5}{15} = \frac{20}{60}$, or $5 : 15 = 20 : 60$, or $5 : 15 :: 20 : 60$. The first and fourth are called the extreme terms, and the second and third the mean terms. The last is the ordinary method of expressing proportion, and the sign :: has the same meaning as the sign $=$.

316. The following are examples of proportions:—

(a). 21 : 7 :: 30 : 10
(b). 7 : 9 :: 42 : 54

In (a) the common ratio is 3; that is, the number of times 21 contains 7, is $\frac{21}{7}$ or 3; so also is the number of times 30 contains 10. In (b) the common ratio is $\frac{7}{9}$. In both cases there is the same ratio between the first and second as between the third and fourth.

317. *Observation.*—Although it is necessary that the two terms of a ratio, when they are concrete numbers, should refer to the same denomination, it is not necessary that all the four terms of a proportion should be of the same name. Thus there is the same ratio between £5 and £35 as between 2 months and 14 months. Hence £5 : £35 :: 2 months : 14 months. Five pounds *are to* thirty-five pounds *as* two months *are to* fourteen months. This is a true proportion: for we do not assert equality between money and time; we only assert equality of *relationship* between 2 sums of money and 2 times; that is, the mutual relation of the two sums of money is the same as that of the two periods of time. But when numbers are once arranged in this way, it is often necessary, in practice, to disregard their special signification, and to treat them as abstract numbers.

The following is a brief investigation of the conditions under which abstract numbers are in proportion.

318. * *When four numbers are so related that any equi-multiples of the first and third can be found equal respectively to any other equi-multiples of the second and fourth, these four numbers are in proportion.*

Demonstrative Example.—4, 7, 12, 21. Here if we multiply the first and third terms by 14, we obtain 56 and 168; but on multiplying 7 and 21 by 8, we also obtain 56 and 168. Therefore 4 has the same relation to 7 as 12 to 28.

For because 14 times the first term equals 8 times the second, the first $= \frac{8}{14}$ of the second, or has to the second the ratio of $8 : 14$; but in like manner, 14 times the third equals 8 times the fourth, and the third $= \frac{8}{14}$ of the fourth, or is to the fourth as $8 : 14$. Wherefore the ratio of first to second = that of third to fourth.

General Formula.—If a b c d be such numbers that $ma = nb$, and $mc = nd$, then $\frac{a}{b} = \frac{c}{d}$ or $a : b = c : d$.

319. † *Whenever four numbers are in proportion, the product of the two extreme terms is equal to the product of the two mean terms.*

Demonstrative Example I.—This is proved by (126). For by the definition of proportion, if the first be greater than the second, the fourth is exactly as many times less than the third; and if the first be less than the second, the fourth is as many times greater than the third. Wherefore the product of the second and third must equal that of the first and fourth.

* In the introduction to the fifth book of Euclid, there is a definition of proportion, which resembles the statement here made, but is somewhat more comprehensive. The reason for Euclid's wider definition is that, in geometry, proportion has often to be asserted of quantities which are incommensurable, or whose ratio cannot be numerically expressed. Thus the side of one square is to its diagonal as the side of a greater square is to its diagonal; but since the ratio of the side to the diagonal of a square is not expressible by numbers, no multiple of a side could ever be found equal to a multiple of a diagonal. Yet there certainly exists a proportion or equality of ratios between the first two magnitudes and the second. Euclid's definition is intended to meet such cases; but while proportion of numbers only is considered, the statement given in the text is sufficient, and is invariably true.

† *Proposition xiv., Book vii., Euclid.*

Demonstrative Example II.—In the proportion 48 : 16 :: 27 : 9, let us multiply the two consequents by the common ratio 3; then the proportion becomes 48 : 48 :: 27 : 27. But as one of the extremes and one of the means have been multiplied by the same number, the product of the extremes and the product of the means have been also multiplied by the same number. But it is evident that in the second case the products are equal; therefore they were equal before both were multiplied by the same number.

Demonstrative Example III.—5 : 9 :: 20 : 36. Here the common ratio is $\frac{5}{9}$. Multiply both the second and fourth terms (one extreme and one mean) by $\frac{5}{9}$, and the extremes will become the same as the means. Their products therefore were equal before they were multiplied by the same fraction.

General Formula.—If $a : b :: c : d$, then $ad = bc$;

For (313) if $a : b :: c : d$, $\frac{a}{b} = \frac{c}{d}$.

Multiply both by bd, then $\frac{abd}{b} = \frac{cbd}{d}$.

Reduce them to their lowest name, $ad = cb$.

320. TO FIND ANY TERM OF A PROPORTION IF THE REMAINING THREE ARE KNOWN—

RULE.

If the required be a mean term, find the product of the extremes and divide by the remaining mean; but if the required term be an extreme, find the product of the mean terms, and divide by the given extreme.

For if $a : b :: c : d$; then (319) $ad = cb$.

Hence (125) $a = \frac{cb}{d}$, $d = \frac{cb}{a}$, $c = \frac{ad}{b}$, $b = \frac{ad}{c}$.

Example.—What is the third term of a proportion of which the first is 150, the second 80, and the fourth 12?

Here because 150 : 80 :: the third term : 12;

By (125) the third term $= \frac{150 \times 12}{80} = 22{\cdot}5$.

Hence 150 : 80 = 22·5 : 12.

EXERCISE XCIX.

Fill up the vacant places in the following proportions with suitable numbers :—

1. 7 : 42 :: ( ) : 120.
2. 28 : ( ) :: 100 : 25.
3. ( ) : 17 :: 805 : 67.
4. 12 : 58 :: 342 : ( ).
5. ( ) : 79 :: 683 : 15.
6. 27 : ( ) :: 108 : 97.
7. 832 : 122 :: ( ) : 2440.
8. ( ) : 50 :: 9 : 1.
9. 264 : 1000 :: 66 : ( ).
10. 287 : 372 :: ( ) : 5376.
11. What is the number which has the same ratio to 17 as 43 has to 4·25?
12. What number has the same ratio to 187 as the difference between 8 and 3 bears to their sum?

321. *Whenever the product of two extreme terms equals that of the two mean terms, the four numbers so arranged are in proportion.*

Demonstrative Example.—Because $3 \times 12 = 4 \times 9$, therefore 3 : 4 :: 9 : 12. For it is evident that if the number in the fourth place, for example, when multiplied by the first, equals the product of the second and third; this fourth number must be as much greater or less than the third as the second is greater or less than the first; and such numbers (315) are always in proportion.

General Formula.—If $ab = cd$, then dividing both by cb, $\frac{ab}{cb} = \frac{cd}{cb}$.

Reducing them to the lowest name, $\frac{a}{c} = \frac{d}{b}$, or $a : c = d : b$.

322. *Corollary.*—*Whenever the product of two numbers equals the product of two others, and the four numbers are so arranged that one pair of factors shall form the two extremes, and the other the means, they shall be in proportion.*

If $6 \times 12 = 9 \times 8$—	If $ad = cb$—
Then 6 : 9 :: 8 : 12	$a : b :: c : d$
12 : 9 :: 8 : 6	$c : a :: d : b$
9 : 6 :: 12 : 8	$d : c :: b : a$
6 : 8 :: 9 : 12	$a : c :: b : d$

EXERCISE C.

Arrange the terms of each of the proportions in the last Exercise in as many ways as you can, and make six other examples.

323. *If two numbers be prime to each other, no smaller numbers can be found which have the same ratio.*

Demonstrative Example.—Because 4 and 9 are prime to one another, the ratio 4 : 9, or $\frac{4}{9}$, cannot be expressed by any smaller numbers than 4 : 9. For if it were possible, then both antecedent and consequent could be divided by some one number, and the two quotients would represent the ratio between them. But this one number would necessarily be a common divisor of 4 and 9. Whereas by the hypothesis they have no common measure.*

It is always convenient to express a given ratio in its lowest terms.

Exercise CI.

(a). Express the following ratios in their lowest terms :—

1. 14 : 77 ;   200 : 384 ;   616 : 7056 ;   63 : 135.
2. 936 : 2368 ;   81 : 4872 ;   220 : 528 ;   35 : 315.
3. 4067 : 2573 ;   856 : 936 ;   1242 : 2323 ;   176 : 2000.

(b). Make 12 different proportions, and show—1. That equi-multiples of the first and third can be found equal to other equi-multiples of the second and fourth. 2. That the product of first and fourth equals that of second and third ; and 3. That each proportion may be stated in four different ways.

324. When the same number is the consequent of one ratio and the antecedent of the next, the three numbers are said to be in Continued Proportion.

Thus 7 : 21 : : 21 : 63. This is usually written thus—7 : 21 : 63. In such a case it follows from (124) that the product of the extremes equals the square of the mean, thus $7 \times 61 = 21 \times 21 = 21^2$. It is also obvious that either of the extremes may be found if the other two are given, by squaring the mean term and dividing this square by the other extreme. Thus $7 = \frac{21^2}{63}$ and $63 = \frac{21^2}{7}$.

325. Sometimes more than 3 numbers are in continued proportion. Thus 5 : 15 : 45 : 135 : 405.

Here $\frac{1}{3}$ is the common ratio subsisting between every pair of adjacent numbers ; and $\frac{5}{15} = \frac{15}{45} = \frac{45}{135} = \frac{135}{405}$.

Series of this kind are said to be in Geometrical Progression, and are investigated in the section on that subject.

* Euclid, Book vii., prop. 23.

SECTION II.

The following propositions are true in all cases of proportion :—

326. *I.—If four numbers be in proportion, the sum or difference of the first and second is to the second, as the sum or difference of the third and fourth is to the fourth.*

If $20 : 12 :: 5 : 3$; then $20 \pm 12 : 12 :: 5 \pm 3 : 3$.

Let $w : x :: y : z$; then (313) $\frac{w}{x} = \frac{y}{z}$. Add 1 to both equals; then $\frac{w}{x} + 1 = \frac{y}{z} + 1$. But $1 = \frac{x}{x}$ or $\frac{z}{z}$. Therefore $\frac{w}{x} + \frac{x}{x} = \frac{y}{z} + \frac{z}{z}$, or $\frac{w + x}{x} = \frac{y + z}{z}$. Hence $w + x : x :: y + z : z$.

Again subtract 1 from both sides of $\frac{w}{x} = \frac{y}{z}$. Then $\frac{w}{x} - \frac{x}{x} = \frac{y}{z} - \frac{z}{z}$, or $\frac{w - x}{x} = \frac{y - z}{z}$, or $w - x : x :: y - z : z$.

Hence $w \pm x : x :: y \pm z : z$.

327. *II. If there be two equal ratios, the sum or difference of the antecedents is to the sum or difference of the consequents, as either of the antecedents is to its consequent.*

If $100 : 25 :: 16 : 4$, then $100 \pm 16 : 25 \pm 4 :: 100 : 25 :: 16 : 4$.

Let $w : x :: y : z$; then $\frac{w}{x} = \frac{y}{z}$. Also by (322) $\frac{w}{y} = \frac{x}{z}$. But (326) if $\frac{w}{y} = \frac{x}{z}$, then $\frac{w \pm y}{y} = \frac{x \pm z}{z}$, or $w \pm y : y :: x \pm z : z$.

Hence by (322) $w \pm y : x \pm z :: y : z :: x : y$.

328. *Corollary.*—The sum of the antecedents is therefore to the difference of the antecedents, as the sum of the consequents is to the difference of the consequents. For these ratios are each equal to the original ratio. Hence they are equal to one another.

329. *III.—If there be any number of equal ratios, the sum of all the antecedents is to the sum of all the consequents, as either of the antecedents is to its consequent.**

$$3 : 5 :: 9 : 15 :: 18 : 30 :: 330 : 550.$$

$$\therefore \frac{3 + 9 + 18 + 330}{5 + 15 + 30 + 550} = \tfrac{3}{5} = \tfrac{9}{15} = \tfrac{18}{30} = \tfrac{330}{550}.$$

* Euclid, Book vii., 12.

Let $a : b :: c : d :: e : f$, then $\frac{a+c+e}{b+d+f} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$. For by (326) if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+c}{b+d} = \frac{a}{b}$. But $\frac{a}{b} = \frac{e}{f}$. Wherefore $a + c : b + d :: e : f$. The sum of these antecedents and consequents may again be taken, when $\frac{a+c+e}{b+d+f} = \frac{a}{b} = \frac{e}{f}$.

330. *IV.—If any number of pairs of equal ratios be taken and arranged in like order, and all the antecedents on each side be multiplied for new antecedents, and also all the consequents on each side for new consequents, the four products thus obtained will also be in proportion.*

If $4 : 7 :: 12 : 21$; $5 : 18 :: 120 : 432$; and $13 : 9 :: 39 : 27$; then $4 \times 5 \times 13 : 7 \times 18 \times 9 :: 12 \times 120 \times 39 : 21 \times 432 \times 27$.

If $a : b :: c : d$
and $l : m :: n : r$
and $w : x :: y : z$

Then $alw : bmx :: cny : drz$.

For if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} \times \frac{l}{m} = \frac{c}{d} \times \frac{n}{r}$, because by hypothesis $\frac{l}{m} = \frac{n}{r}$; and we have therefore only multiplied two equals by the same ratio. So because $\frac{al}{bm} = \frac{cn}{dr}$, $\frac{al}{bm} \times \frac{w}{x} = \frac{cn}{dr} \times \frac{y}{z}$.

331. *Corollary.*—If four numbers are in proportion their squares are in proportion, also their cubes, their fourth powers, &c.

For it is evident that if the proportion $a : b :: c : d$ be compounded with itself by multiplication, as described in (330), the result will be $\frac{a^2}{b^2} = \frac{c^2}{d^2}$, or $\frac{a^n}{b^n} = \frac{c^n}{a^n}$.

By that alternation of the order of the several terms, which is allowed by (322), the results of the four last propositions may be varied in several ways.

EXERCISE CII.

Verify each of the last four propositions by the proportions given in Exercise XCIX., and by 12 other proportions of your own selection.

Example.—Because $8 : 5 :: 24 : 15$.

By 326. $13 : 5 :: 39 : 15$, and $3 : 5 :: 9 : 15$.

By 327. $32 : 20 :: 8 : 5$, and $16 : 10 :: 24 : 15$.

By 328. $13 : 3 :: 39 : 9$.

By 331. $64 : 25 :: 576 : 225$.

SECTION III.—THE RULE OF THREE; OR, SIMPLE PROPORTION.

332. The Rule by which the principles of Proportion are applied to Practice is called the Rule of Three.

It is so called because in all the questions proposed in it *three* terms of a proportion are given, and it is required to find a fourth.

333. *Observation.*—The doctrine of proportion affects more or less every part of arithmetic. In many of the processes we have already employed the existence of proportion among the several numbers has been assumed. Thus in Compound Multiplication, if we ask, What will be the price of 79 articles if one costs 8s. 6d.? it is assumed that there is to be the same ratio between the number of articles as between the prices paid for them. So when in Division the question occurs, "If 20 men dig 45 acres of ground, what will 1 dig?" it is assumed that the quantity of work done will be proportioned to the number of workers; and we are asked to find a number of acres bearing the same proportion to 45 as 1 bears to 20.

334. In every sum in Simple Proportion two things are required:—

I. To determine in what relation the unknown number is to stand to the given number of the same denomination.

II. To find a number which shall fulfil the required conditions.

335. The former of these is in fact a *theorem* in Proportion, and requires to be thought out by the help of the principles laid down on that subject. The latter is merely a *problem* in Multiplication and Division. For when three terms of a proportion are known and arranged the fourth may be readily found by (320), because in all such cases the product of the extremes equals the product of the means. But before that rule can be employed it is necessary to arrange the terms, and to do this we must first discover in what relation the required answer must stand to a certain known quantity, and then lay out the given terms in such a way as to represent that relation. This task of *arranging the terms* forms the principal difficulty of the Rule of Three.

336. *Direct Proportion.*—When any two magnitudes are so related that as the one increases or diminishes the other increases or diminishes, they are said to vary *directly*, or to be in *Direct Proportion* the one to the other.

Example.—If a number of men are employed upon a certain work, the more men are hired the more work will be done, and the less the work to be done the less men will be required. Hence the number of men and the quantity of work are in *direct proportion*, or the one number *varies directly* as the other; for to increase or diminish the one is to increase or diminish the other.

337. In the same way it appears that, all other things being equal—

The price of any number of articles varies as the number of articles.
The cost of conveyance varies as the distance travelled.
The rent of land varies as the time of the tenure.
The wages of a labourer vary as the time of his employment.
The interest paid for the use of money varies as the sum lent.
The taxes paid by a householder vary as the rental of his house.
Work done in a given time varies as the number of agents.

For let a and b be any numbers; then—

The price of a articles : the price of b articles :: a : b.

The cost of carrying luggage a miles : the cost of carrying it b miles :: a : b.

The rent of land for a years : the rent for b years :: a : b.

Labourers' wages for a weeks : wages for b weeks :: a : b.

Interest paid upon £a : interest paid on £b :: a : b.

Taxes paid on a house rated at a : taxes paid on a house rated at b :: a : b.

Work done by a men : work done by b men :: a : b.

Observation.—In every one of these cases there is the same ratio between two magnitudes of one kind as between the two corresponding terms of another kind. The term relating to the number a is always in the first place and forms the antecedent of a ratio, while the number a itself is always in the third term and forms the antecedent of another ratio. Thus in every case the two antecedents belong to one another in the same way as the two consequents, and the proportion is said to be *direct*.

338. *Inverse Proportion.*—When any two magnitudes are so related that as the one increases the other diminishes, or as the one diminishes the other increases, they are said to vary *inversely*, or to be in Inverse or *Indirect Proportion* the one to the other.

Example.—If a given work has to be performed and a number of men is hired to do it, the greater the number of men the less time will be occupied, and the greater the time allowed the fewer men will be required. Here the number of men required is in *inverse* proportion to the time employed; to increase the one is to diminish the other and *vice versâ.*

339. In like manner it appears that, all other things being equal—

Length of pieces (of carpet, &c.) required to cover a surface varies inversely as the breadth of the stuff.

Number of coins required to pay a certain sum varies inversely as the value of the coins.

Velocity of a body moving through a space varies inversely as the time occupied in its passage.

Weight of luggage carried for a given sum varies inversely as the distance of conveyance.

For let a and b again represent any numbers; then—

Length of carpet a feet wide required to cover a certain floor : length of carpet b feet wide to cover the same :: b : a.

Number of coins value a required to pay a certain sum : number value b to pay the same :: b : a.

Velocity of a body traversing a certain distance in a seconds : velocity of a body traversing the same distance in b seconds :: b : a.

Weight of luggage carried a miles for a certain sum : weight carried b miles for the same sum :: b : a.

Observation.—In every one of these cases the increase of one term in a ratio makes it necessary to diminish the corresponding term in the other ratio, and the two kinds of number are said to be *reciprocally* or *inversely* proportional the one to the other, the terms of the second ratio being *inverted* with respect to the first. Here it will

be seen that if the term belonging to the number a is in the first place and forms the antecedent of one ratio, the number a itself is in the *fourth* place and forms the consequent of the other ratio.

The meaning of direct and inverse proportion becomes further evident when we consider the nature of a fraction, thus—

340. *If two fractions have the same denominators, they are to one another as their numerators*, i.e., *they vary directly as their numerators.*

Demonstrative Example.—$\frac{5}{16} : \frac{7}{16} :: 5 : 7$.

For it has already been shown (189) that fractions having the same denominators are greater or less according as their numerators are greater or less, and it is obvious that the product of the first and fourth ($\frac{5}{16} \times 7$) equals the product of the second and third ($\frac{7}{16} \times 5$). Hence the four terms make a proportion (321).

The fraction $\frac{5}{16}$ and the 5 which corresponds to it are the two antecedents, and $\frac{7}{16}$ and 7 are the two consequents. The proportion is therefore *direct.*

General Formula.—$\frac{a}{x} : \frac{b}{x} :: a : b$; for $\frac{a}{x} \times b = \frac{b}{x} \times a$.

341. *But if two fractions have the same numerator but different denominators, they are to one another in the inverse ratio of their denominators*, i.e., *they vary inversely as their denominators.*

Demonstrative Example.—$\frac{7}{9} : \frac{7}{12} :: 12 : 9 = \frac{1}{9} : \frac{1}{12}$.

For it has already been shown (190) that with the same numerator a fraction is greater as its denominator is less, and less as its denominator is greater. It is also true that $\frac{7}{9} \times 9 = \frac{7}{12} \times 12$. Wherefore fractions vary inversely as their denominators.

General Formula.—$\frac{a}{x} : \frac{a}{y} :: y : x = \frac{1}{x} : \frac{1}{y}$.

342. If we bear in mind the connexion between the formulæ of Fractions and of Division, the last statements will appear to be equivalent to the following :—

If two Division sums have the same divisor but different dividends, the quotients vary directly as their dividends.

But if two Division sums have the same dividends but different divisors, the quotients vary inversely as their divisors.

343. The reasoning required in every Rule of Three sum takes the following form. We must—

I. Ascertain of what kind or denomination the answer is to be.

II. Select that one of the three given terms which relates to the same kind as the answer.

III. Make this term and the unknown term the antecedent and consequent of a ratio.

IV. Make the remaining terms the antecedent and consequent of another ratio, arranged so that the proportion may be true.

344. When the terms have been thus arranged, the solution of the question depends on the principle already explained (319), and on the rule which is founded on it, viz:—

Take the product of any pair, either of extreme or mean terms, and divide it by the remaining one; the quotient will be the fourth proportional required.

Observation.—The denomination of the number thus found is always the same as that which forms its antecedent or consequent.

345. *Example I.*—Suppose it is required to find the cost of 113 lbs. when 7 lbs. cost 20 pence.

Here the price of a certain weight is given and it is required to find the price of another weight. But because price is directly proportioned to the quantity purchased, the two weights must have the same ratio as the two sums of money.

Hence if x equals the unknown sum of money—

$$20\text{d.} : x\text{d.} :: 7 \text{ lbs.} : 113 \text{ lbs.}$$
$$113 \text{ lbs.} : 7 \text{ lbs.} :: x : 20.$$
$$x : 20 :: 113 : 7.$$
$$7 : 113 :: 20 : x.$$

Each of these proportions must be true; for in all, the two weights form the antecedent and consequent of one ratio, and the two prices (*i.e.*, the known and the unknown sum of money) form the corresponding antecedent and consequent of the other ratio.

In each of these cases, therefore, the missing term is to be found by multiplying 113 by 20 and dividing by 7, or—

$$x = \frac{113 \times 20}{7} = \overset{d.}{322 \cdot 58} = \overset{£.}{1} \quad \overset{s.}{6} \quad \overset{d.}{10 \cdot 58}.$$

Observation.—It was not necessary to consider any of the numbers as concrete while the sum was being worked, for all that was required was to find a number which should bear the same relation to 7 as 113 did to 20. But this number when found is of the same denomination as the 7—*i.e.*, it is a number of pence, and the answer has to be reduced to pounds.

346. *Example II.*—25 men reap 60 acres in a certain time, how many acres will 4 men reap in the same time?

Here we want to find a certain number of acres, let it be x.

But 60 is a given number of acres, and these two numbers have a ratio one to the other—$60 : x$.

But (337) work done is directly proportioned to the number of agents.

∴ as the number of men who reap 60 acres is to the number who reap x, so is 60 to x.

$$\text{Hence } 25 : 4 :: 60 : x.$$
$$\text{Or } 60 : x :: 25 : 4.$$
$$\text{Or } x : 60 :: 4 : 25.$$
$$\text{Or } 4 : 25 :: x : 60.$$

We want a number of acres which shall be as many times less than 60 as 4 is less than 25, and either of these proportions will give us such a number; for—

$$x = \frac{4 \times 60}{25} = 9 \cdot 6 = 9 \cdot 6 \text{ acres}.$$

347. *Example III.*—If 120 yards of carpet 3 qrs. wide will cover a floor, how many yards would be required if the carpet were 5 qrs. wide?

Here the answer (x) will require to be the length of carpet.

But 120 is the given term which represents length of carpet.

∴ $120 : x$ is the ratio to be solved.

But the wider the carpet is the less length will be required, for (339) with a given surface length and breadth are *inversely* proportioned to each other.

Hence x must be less than 120 as many times as 5 is greater than 3.

$$x : 120 :: 3 : 5.$$
$$\text{Or } 3 : 5 :: x : 120.$$
$$\text{Or } 120 : x :: 5 : 3.$$
$$\text{Or } 5 : 3 :: 120 : x.$$

In every case $\frac{120 \times 3}{5} = 72$ yards, answer required.

348. From (311) it appears that if two numbers are not of the same denomination they do not represent the ratio of the magnitudes which they express. Hence if the two terms of any ratio are not in the same name they must be made so before the sum can be worked. This can always be done by reducing the numbers of the higher into those of the lower denomination.

349. *Example IV.*—If 2 tons 9 cwt. cost £114 16s., what will 26 lbs. cost?

Here a sum of money is the answer required (x).

But £114 16s. is a sum of money.

∴ £114 16s. : x is the ratio required.

But (337) the quantity purchased is directly proportioned to the sum spent.

Hence, as the price of 2 tons 9 cwt. is to the price of 26 lbs, so is 2 tons 9 cwt. to 26 lbs., or £114·8 : x :: 2 tons 9 cwt. : 26 lbs. But the figures in this statement do not represent the ratio properly because they are concrete numbers of different denominations.

Wherefore reducing 2 tons 9 cwt. to pounds, and £114 16s. to the decimal of £1—

$$x : 114{\cdot}8 :: 26 : 5488.$$
$$114{\cdot}8 : x :: 5488 : 26.$$
$$26 : 5488 :: x : 114{\cdot}8.$$
$$5488 : 26 :: 114{\cdot}8 : x.$$

In each case the answer will be found in the same manner.

$$x = \frac{26 \times 114{\cdot}8}{5488} = \pounds{\cdot}54 = 10\text{s. } 9\tfrac{1}{4}\text{d.}$$

350. Sometimes a number is mentioned in the question which does not affect the answer and need not be stated.

Example V.—If I can have 3 cwt. 2 qrs. carried 80 miles for 18s., how far can I have 2 tons carried for the same money?

Here the 16s. has nothing to do with the question, for if any other sum were mentioned the answer would be just the same.

The answer is to be a certain distance, and 80 miles is a number of the same kind as the answer.

$\therefore$ 80 : x is the ratio to be determined.

But (339) the distance is inversely proportioned to the weight; *i.e.*, the greater the weight the less distance it will be carried for a given sum.

$\therefore$ the answer will be less than 80 miles, and the consequent of the first ratio must be less than its antecedent.

$\therefore$ 2 tons : 3 cwt. 2 qrs. :: 80 : x.

Reducing the first two terms to the same name—

$$\begin{array}{rrrr} 160 \text{ qrs.} : & 14 \text{ qrs.} :: & 80 : & x. \\ \text{Or } 80 : & x :: & 160 : & 14. \\ 14 : & 160 :: & x : & 80. \\ x : & 80 :: & 14 : & 160. \end{array}$$

In all the cases—

$$x = \frac{80 \times 14}{160} = 7 \text{ miles}.$$

351. *Example VI.*—A father divides £1280 among his 3 children, so that their portions are as 5, 3, and 2 respectively; how much does each receive?

Here are three several sums to be worked, as we want three answers. We first notice that, as the answers will be in money, £1280 must be the third term in each case.

Now $5 + 3 + 2 = 10$. Wherefore the first is to have £5 out of every £10, the second £3, and the third £2.

$\therefore$ 10 : 5 :: 1280 : Share of the first.
And 10 : 3 :: 1280 : Share of the second.
And 10 : 2 :: 1280 : Share of the third.

$\therefore$ the three answers are £640, £384, and £256, respectively.

352. *Example VII.*—£4000 have to be divided among 4 persons in the proportions of $\frac{1}{6}$, $\frac{2}{5}$, $\frac{4}{9}$, and $\frac{1}{3}$; what is the share of each?

Reducing these fractions to a common denominator, we find them

$\frac{15}{90}$, $\frac{36}{90}$, $\frac{40}{90}$, $\frac{30}{90}$, in all $\frac{121}{90}$. The 90 does not affect the question, but the whole must be divided into 121 parts.

$$\begin{aligned} \therefore\quad 121 : 15 :: 4000 &: \text{Share of the first.} \\ 121 : 36 :: 4000 &: \text{Share of the second.} \\ 121 : 40 :: 4000 &: \text{Share of the third.} \\ 121 : 30 :: 4000 &: \text{Share of the fourth.} \end{aligned}$$

The four answers will be found to make £4000 if added together, and to be to one another in the proportions of $\frac{1}{6}$, $\frac{2}{5}$, $\frac{4}{9}$, and $\frac{1}{3}$.

353. In the examples just given we have placed the symbol representing the required answer, sometimes in one place of the proportion and sometimes in another. It is important to notice that any one of these places (first, second, third, or fourth) is just as suitable as any other, provided the remaining terms be so placed as to form a true proportion. Nevertheless, it is usual *to reserve the fourth place for the answer in all cases*, and to arrange the remaining terms according to the following rule.

RULE OF THREE.

354. Place the term which is of the same kind as the required answer in the third place, leaving the fourth place for the answer.

If the answer will be greater than the third term, make the second term greater than the first; but if the fourth is to be less than the third, then place a less term in the second place than in the first.

If the first and second terms be not of the same denomination reduce the higher to the same name as the lower (84).

Then multiply the second and third terms together, and divide by the first.

The quotient thus found is of the same name as that to which the third term has been reduced.

EXERCISE CIII.

Work the following sums :—

1. How many yards of cloth, worth 18s. 3½d. per yard, must be given in exchange for 937½ yards, worth 3s. 7½d. per yard?*

2. If an insolvent, whose whole possessions amount to £1728, pays at the rate of 4s. 3d. in the pound, what does he owe?

3. If an insolvent, whose debts amount to £2740, pays £67 to a creditor whose account is £162 15s., what is the total value of his effects?

4. If £1250 will purchase 2½ tons, what weight can be bought for £2 3s. 9d.?

5. How many coins, worth 2s. 3d. each, will be equal to 10000 German kreutzers, three of which are worth an English penny?

6. Two numbers together make 1800, and they are to one another as 2 to 7; what are they?

7. A man divides £16 among four people, so that for every 10d. one has, another has 9d., another 8d., and the fourth 7d.; how much did each receive?

8. If $\frac{5}{7}$ of a pound cost $\frac{2}{3}$ of a shilling, what will $\frac{2}{3}$ of a cwt. cost?

9. The ratio of the diagonal to the side of a square is about as 99 to 70; what is the diagonal of a square whose side is 3·5 miles? and what is the side of a square whose diagonal is 172·3 yards?

10. If 7 hogsheads, each weighing 3 cwt. 2 qrs. 17 lbs., can be purchased for £54, what must the goods be sold at per lb. to gain £15 on the whole transaction?

* Teachers will find it an excellent exercise, to verify the answers to these sums, by requiring pupils to make three other questions, each of which shall require one of the first named terms as the answer. By this means each term of the proportion will be brought successively into the fourth place, and will be the answer to a sum composed of the other three. Thus, after finding the answer to the first exercise, the proportion admits of being stated in three other ways.

I. If $371\frac{152}{439}$ yards are given in exchange for 937½, which are worth 3s. 7½d. per yard, what is the price per yard?

$$937\tfrac{1}{2} : 371\tfrac{152}{439} :: 3s.\ 7\tfrac{1}{2}d. : x.$$

II. How many yards, worth 3s. 7½d., must be given in exchange for $371\frac{152}{439}$, worth 18s. 3½d.?

$$3s.\ 7\tfrac{1}{2}d. : 18s.\ 3\tfrac{1}{2}d. :: 371\tfrac{152}{439} : x.$$

III. If 937½ yards are given in exchange for $371\frac{152}{439}$ yards, at 18s. 3½d. per yard, what is the price per yard?

$$937\tfrac{1}{2} : 371\tfrac{152}{439} :: 18s.\ 3\tfrac{1}{2}d. : x.$$

11. A bankrupt owes £7250, and his entire property amounts to £587 10s.; what dividend can he pay?

12. Find a fourth proportional to ·0004, 1·4, and ·02; and also to $\frac{2}{7}$, $\frac{3}{4}$, and $\frac{5}{6}$.

13. Find the difference between the cost of 37 cwt. 2 qrs. 14 lbs. at £7 10s. 9d. per cwt., and that of 39 cwt. 3 qrs. 26 lbs. at £4 17s. 10d. per cwt.

14. If a man's wages are regulated by the price of provisions, and he receives 1s. 3d. a day when corn is 7s. per bushel; what will he receive when corn is 5s. 6d. per bushel?

15. If a piece of cloth measuring 9 ells 1 nail 1·125 inches cost £4 10s. 6½d., what is the price per yard?

16. Two couriers pass through a town at an interval of 4 hours, travelling at the rate of 11½ and 17½ miles an hour; how far and how long must the first travel before he is overtaken by the second?

17. Gravity varies directly as the mass and inversely as the square of the distance. Compare the amount of the earth's attraction on two bodies, the one having a mass 35 at a distance 6, and the other having a mass 125 at a distance 11.

18. How many yards of lace can I buy for £228 12s. 7d., at the rate of £1 15s. for 4¼ yards?

19. If I buy 7½ dozen of champagne at £3 5s. per dozen, and 13 dozen 7 bottles of port at 42s. per dozen, what is the amount of my bill?

20. If 2 cwt. 1 qr. 17 lbs. cost £50 11s. 4d., how much can be bought for £18 7s. 6d.?

21. If a street 3 quarters of a mile long be repaired at a cost of £7 9s. 6d., what portion of the expense should be paid by an inhabitant whose premises have a frontage of 18 yards 2 feet?

22. How many reams of paper could be bought for £120, at the rate of £15 6s. for 13 reams.

23. If a train runs to Sydenham, a distance of 5½ miles, in 12 minutes 5 seconds, how long should it take to reach a place which is 49 miles 3 furlongs distant?

24. The areas of circles are as the squares of their diameters. Suppose there are two circles whose diameters are as 11 to 17·5, and the area of the first contains 3 acres 2 roods 17 poles; what is the *area of the second?*

25. If a tradesman who owes £1850 is only worth £567 10s., what ought he to pay in the pound, and how much will a creditor receive whose claim is £58 15s.?

26. If a tradesman, whose debts amount to £2653 10s. 6d., pays 6s. 3½d. in the pound, what is the worth of his effects?

27. If a bankrupt, whose whole possessions amount only to £625 13s., pays a dividend of 1s. 3¼d. in the pound, what does he owe?

28. If I lend £112 for 9 months, how much should the borrower lend me for 6 months, in order to repay the loan?

29. Find the fourth term of the following proportion :—

$$40{\cdot}5 : \overline{3^2 + 5^2 - 2} :: \overline{3^5 \div 3^2} : (\quad).$$

30. If 49 yards of carpeting 34 inches wide, cover a floor, how much will be needed of carpet which is only 29 inches broad?

31. The French "aune" is to the English yard as 1000 to 769. What is the length in French measures of 123 yards 2 feet?

32. The annual value of rateable property in the county of Middlesex is £7584668, what would a tax of 2¾d. in the pound produce if levied on the whole?

33. Find the cost of 39 cwt. 3 qrs. 26 lbs., at £4 17s. 10d. for 2 cwt. 1 qr. 19 lbs.?

34. If a tower 45 feet high has a shadow of 39 feet, what is the height of an obelisk having at the same time a shadow 76·5 feet long?

35. If a servant whose yearly wages are 25 guineas, enters service on the 27th of August, and quits it on the 13th of the following January, what ought he to receive?

36. If after paying income-tax at 1s. 2d. in the pound, a gentleman has £701 10s. 10d. remaining, what is his annual income?

37. If 12·35 lbs. are sold for 7·4 shillings, what quantity could I buy for £253·67?

Exercise CIV.

State each of the last six sums in four ways, making the answer (x) in the first, the second, the third, and the fourth terms of the proportion, as in the examples.

Take the first six sums, and make from each of them three other questions—1, so that the first term shall come in the last place and be the answer; 2, so that the second; and 3, so that the third shall be the answer. See Example (page 200).

SECTION IV.—COMPOUND PROPORTION; OR, THE DOUBLE RULE OF THREE.

355. When the antecedents of two or more ratios are multiplied together and also their consequents, the ratio thus produced is said to be *compounded* of the others.

Thus if we take the two ratios, 7 : 9 and 3 : 4, and multiply their antecedents and consequents so as to get 21 : 36, this is the ratio compounded of 7 to 9 and of 3 to 4.

356. Composition of ratios is exactly equivalent to the multiplication of fractions.

$$\text{Thus } \frac{5}{7} \text{ of } \frac{7}{12} = \frac{35}{36}, \quad \text{or} \left.\begin{array}{rcr} 5 & : & 6 \\ 7 & : & 12 \\ \hline 35 & : & 36 \end{array}\right.$$

357. In solving many sums in Proportion it is necessary to compound one ratio with another in this way. Whenever this is done the sum is said to be in Compound Proportion, or the Double Rule of Three.

The employment of this Rule depends on the following principle:—

358. *If there be three quantities so related that when all other things are equal the first varies as the second, but when all other things are equal the first varies also as the third; the variation of the first is as the product of the second and third.*

Demonstrative Example.—If work done varies as the time of doing it, and also as the number of persons employed; it varies altogether as the product of the number of days, and the number of labourers.

For if a certain work is done by one man in one day, 3 men in 4 days will do 3 × 4, or 12 times as much.

Hence the work done is affected by two circumstances, the time and the number of labourers. But as the time is increased 4 times, this circumstance increases the work 4 times. Besides this, the number of men is increased 3 times. Therefore, as one circumstance tends to increase the answer in the ratio of 3 to 1 and the other in that of 4 to 1, the total increase arising from both is 3 × 4 : 1, or as 12 to 1.

General Formula.—If a increases as b increases, and also as c increases, then the total increase of a is as bc.

359. Whenever two ratios separately affect a third, their total effect upon it is represented by the ratio compounded of those two, *i. e.*, by the ratio formed by the product of the two antecedents and the product of the two consequents.

360. *Example.*—Suppose one ball weighing 5 pounds, and moving at the rate of 20 feet per second, strikes a wall with a certain force; what is the relative force of the blow given by another ball of 11 pounds weight, and moving 30 feet per second.

Now here the second ball weighs more than the first as much as 11 is more than 5; $\therefore$ as far as the weight is concerned the blow of the second ball will be to that of the first as 11 to 5, or $11 : 5$, or $\frac{11}{5}$.

But the second also moves more quickly. $\therefore$ as far as the rapidity is concerned, the blow of the second will be as much greater than that of the first as 30 is than 20, *i. e.*, $30 : 20$, or $\frac{30}{20}$.

Then because one circumstance tends to increase the answer in the proportion of 11 to 5, and another tends to increase it in the proportion of $30 : 20$; the two together will increase it as $\frac{30}{20} \times \frac{11}{5}$, or $\frac{330}{100} = 3\frac{3}{10}$. Therefore the blow in the second case will be 3·3 times that in the first.

361. *Observation.*—For momentum, or force of blow, is a quantity that varies directly as the weight of the striking body, and also as its velocity; and the general numerical formula for momentum, when the velocity and the weight can both be expressed in figures, is momentum $=$ weight $\times$ velocity.

RULE FOR COMPOUND PROPORTION.

362. Find the number which is of the same kind as the required answer, and place it in the third term as in Simple Proportion.

Take any two terms which are of the same kind, and consider these two with the third as a separate sum in Simple Proportion. State it according to (354).

Then take the other two which are of the same kind, and treat them, with the third term, as another distinct sum in Simple Proportion. State it also according to the same Rule.

Arrange all the separate ratios the one beneath the other; then multiply all the first terms together for a new first term or antecedent, and all the second terms together for a new second term or consequent.

Then multiply this new second term by the third and divide by the first, as in Simple Proportion.

363. *Example I.*—If 13 men earn £7 in 6 days, what will be the wages of 72 men for 21 days?

This sum can be broken into two, thus—

(I.) If 13 men earn £7, what will 72 men earn?

By Rule (354) this will be stated—13 : 72 :: 7 : x.

(II.) If £7 are earned in 6 days, how much will be earned in 21 days?

And by Rule (354) this will be stated—6 : 21 :: £7 : x. The two statements must be thus combined:—

$$\begin{array}{lrcr} & & & & & \pounds & & \pounds \\ \text{Men} & 13 & : & 72 & \multirow{2}{*}{$\Big\}$} :: & 7 & : & x \\ \text{Days} & 6 & : & 21 & & & & \\ \hline & 78 & : & 2162 & :: & 7 & & \end{array}$$

And the answer is $\dfrac{2162 \times 7}{78} = £194{\cdot}025\dot{6} = £194\ 0s.\ 6\tfrac{6}{39}d.$

364. *Observation I.*—While we considered the third term in relation to the number of labourers, we neglected the consideration of the time; and while the third term and the time were considered, we neglected to take into account the number of men. Each question had to be determined on its own merits as a separate sum in Simple Proportion; and when this was done, the two ratios were compounded according to the Rule.

Observation II.—Here are two conditions affecting the amount of the wages to be paid, and each of them tends to increase the answer. First: There are more men to pay; and this would make it necessary to have an amount greater than £7, as much as 72 is greater than 13.

∴ as far as the number of labourers is concerned we want an amount which shall be to £7 as 72 is to 13; *i.e.*, we want $\frac{72}{13}$ of £7.

But the time of their engagement is also greater.

∴ as far as the time is concerned we want an amount as much greater than £7 as 21 is greater than 6; *i.e.*, we want $\frac{21}{6}$ of £7.

Hence the answer we require is neither of them separately, but the combined effect of the two, and we find this by multiplying the two fractions, or compounding the two ratios.

For our fourth term is to be, not $\frac{72}{13}$ of £7 only, nor $\frac{21}{6}$ of £7 only, but $\frac{72}{13} \times \frac{21}{6}$ of £7. And the ratio required is not $13 : 72 :: 7 : x$, nor $6 : 21 :: 7 : x$, but $\left.\begin{matrix} 13 : 72 \\ 6 : 21 \end{matrix}\right\} :: 7 : x$, or $13 \times 6 : 72 \times 21 :: 7 : x$.

365. *Example II.*—If in a certain business £900 profit be realized on an investment of £5200 for 10 months, what should be the profit accruing on £7950 invested for 6 months?

Here the answer is to be *profit;* and £900 represents profit. $\therefore$ £900 is to be the third term, and £900 : x is the ratio to be solved.

There are two separate sums here.

I. If a man with a capital of £5200 realizes £900, what should he realize whose capital is £7950?

By Rule (354) this must be stated—$5200 : 7950 :: 900 : x$.

II. If £900 are realized in 10 months, what should be realized in 6?

By Rule (354) $10 : 6 :: 900 : x$.

Combining these two statements—

$$\left.\begin{matrix} 5200 & : & 7950 \\ 10 & : & 6 \end{matrix}\right\} :: 900 : x$$

$$5200 \times 10 : 7950 \times 6 :: 900 : x$$

$$\therefore x = \frac{7950 \times 6 \times 900}{5200 \times 10};$$

or by cancelling, $x = \dfrac{795 \times 6 \times 9}{52} =$ £825·576 = £825 11s. $6\frac{6}{13}$d.

366. *Observation.*—Here the two circumstances affected the answer differently. The greater amount of capital employed in the second case tended to make the profit greater; but the fact that it was invested for a shorter period tended to make the profit less. Hence the total profit cannot be in the ratio represented by either of these alone, but in a ratio compounded of both. The owner is not to have $\frac{7950}{5200}$ of £900, as he would if the times had been equal; nor is he only to have $\frac{6}{10}$ of £900, as he would have had if the investments had been equal; but he will have $\frac{7950}{5200}$ of $\frac{6}{10}$ of £900, or £$\dfrac{7950 \times 6 \times 900}{5200 \times 10}$.

367. *Example III.*—If 12 horses plough a field of 8 acres in 3 days, in what time will 21 horses plough a field of 100 acres?

Here, because the answer is to be time, and 3 days represents a number of days;

3 days is to be in the third term, and the ratio to be determined is $3 : x$. There are here two sums.

I. If in 3 days 12 horses do any work, in what time will 21 horses do it?

This is a case of Inverse Proportion (338), because as more agents are employed less time will be required.

∴ the statement is $21 : 12 :: 3 : x$.

II. If in 3 days 8 acres are ploughed, in what time will 100 acres be ploughed?

This is a case of Direct Proportion, for as more work has to be done more time will be required.

∴ the statement is $8 : 100 :: 3 : x$.

Combining these two statements—

$$\begin{array}{r} \left.\begin{array}{r} 21 : 12 \\ 8 : 100 \end{array}\right\} :: 3 : x \\ \hline 21 \times 8 : 100 \times 12 :: 3 : x \end{array}$$

$$x = \frac{100 \times 12 \times 3}{21 \times 8} = 21{\cdot}428 \text{ days.}$$

368. *Example IV.*—If 125 men can dig a trench 100 yards long, 20 yards wide, and 4 feet deep, in 4 days, working 12 hours a day; how many men must be engaged to dig another trench 500 yards long, 8 yards wide, and 6 feet deep, in 3 days, working $7\frac{1}{2}$ hours per day?

Here the answer is to be a number of men, and 125 is of the same kind, wherefore $125 : x$ is the ratio to be solved.

Several conditions affect this ratio, and each may be made in turn the subject of a Simple Proportion sum.

I. If 125 men dig a trench 100 yards long, how many will dig one 500 yards long?

By Rule (354) the answer will be greater, the proportion being direct.

∴ the statement is $100 : 500 :: 125 : x$.

II. If 125 men dig a trench 20 yards wide, how many will be required to dig one 8 yards wide?

By Rule (354) the answer to this question will be less than 125, the proportion being direct.

∴ the statement is $20 : 8 :: 125 : x$.

III. If 125 men dig a trench 4 feet deep, how many will dig one 6 feet deep?

By Rule (354) this answer is required to be greater than 125, the proportion being direct.

∴ the statement is $4 : 6 :: 125 : x$.

IV. If 125 men do a certain work in 4 days, how many will do it in 3?

By Rule (354) this answer will require to be greater than 125, the proportion being inverse.

∴ the statement is $3 : 4 :: 125 : x$.

V. If 125 men do a certain work, when employed 12 hours a day, how many will do it when employed $7\frac{1}{2}$ hours a day?

By Rule (354) this answer needs to be greater than 125, the proportion being inverse.

∴ the statement is $7{\cdot}5 : 12 :: 125 : x$.

Combining these statements—

$$\left.\begin{array}{rcr} 100 & : & 500 \\ 20 & : & 8 \\ 4 & : & 6 \\ 3 & : & 4 \\ 7{\cdot}5 & : & 12 \end{array}\right\} :: 125 : x$$

$$\overline{100 \times 20 \times 4 \times 3 \times 7{\cdot}5 : 500 \times 8 \times 6 \times 4 \times 12 :: 125 : x}$$

Or by cancelling—

$$x = \frac{5 \times 8 \times 2 \times 12 \times 125}{20 \times 7{\cdot}5 \times 3} = \frac{125 \times 32}{5} = 800 \text{ men.}$$

By taking all the dimensions of each trench at one statement the sum would have assumed this form—

$$\left.\begin{array}{rcr} 100 \times 20 \times 1{\cdot}3 & : & 500 \times 8 \times 2 \\ 3 \times 7{\cdot}5 & : & 12 \times 4 \end{array}\right\} :: 125 : x$$

$$\text{Cancelling} \quad 5 \quad : \quad 32 :: 125 : x.$$

Observation.—As each of the five circumstances mentioned, the length, the breadth, the depth of the trench, the number of days, and the hours per day, affected the answer differently, each had to be considered separately, and the combined effect was discovered by *compounding* the several ratios one with another.

EXERCISE CV.

1. If it takes 12 yards of cloth $\frac{5}{4}$ wide to make clothes for 500 soldiers, how much cloth $\frac{7}{8}$ wide will be needed to clothe 960 men?

2. A man journeying 15 hours a day, reaches a distance of 375 leagues in 20 days; how many hours per day must he travel to pass over 400 leagues in 18 days?

3. It takes 15 labourers, working 10 hours a day, 18 days to get through 450 yards of a certain work; how many men, working 2 hours a day, would, in 8 days, finish 480 yards of the same work?

4. If 7 horses be kept 20 days for £14, how many may be kept 7 days for £20?

5. If 850 men consume 240 quarters of wheat in six months, in what time would 3230 men consume 1820 quarters?

6. How long will it take 17 men to earn £50, if 12 men, in $6\frac{1}{2}$ days, can earn 13 guineas?

7. If 20 cannon, firing 5 rounds in 6 minutes, kill 500 men in an hour, how many men would be killed in an hour and a half by 10 cannon, firing at the rate of 3 rounds in 5 minutes?

8. If 300 men could do a piece of work in 2·4 days, how many would do one-third of the same work in 12 days?

9. If the work done by a man, a woman, and a boy respectively, be proportioned as 3, 2, and 1; and if 9 men, 15 women, and 18 boys finish a certain work in 208 days, in what time would 15 men, 12 women, and 9 boys finish the same?

10. 250 men are set to work on a railway embankment, a mile and a half long, which they are expected to finish in 4 weeks. But at the end of one week it is found that they have only finished 520 yards; how many more men must be engaged to finish it in the required time?

11. If a mass of silver be worth £720,000 when silver is worth £3 17s. 6d. per pound troy, how much would it be worth when the current price is at the rate of 13·75 shillings for 2·5 ounces?

12. A party of 7 gentlemen, on a journey together, spend £150 in 3 weeks 4 days; what would be the expense, at the same rate, of another party, consisting of 11 persons, travelling for a fortnight?

13. If a plot of building land 272 feet 6 inches long by 35 feet 8 inches broad, be sold for £76 10s. 9d., how much will a plot 283 yards 2 feet long by 74 feet 9 inches wide cost?

SECTION V.—INTEREST, DISCOUNT, ETC.

369. INTEREST is money paid for the use of money.

The money lent is called the PRINCIPAL.

The interest allowed per annum on every £100 of the principal is called the RATE PER CENT.

The sum of the Principal and Interest is called the AMOUNT.

370. Interest is a quantity which *varies directly* as the principal.

It also varies directly as the time for which it is lent.

Whenever principal and interest are alone referred to in a sum, the question is solved by Simple Proportion.

But when principal and time are both considered in relation to the interest, the problem is one in Compound Proportion.

371. As in nearly all problems under this Rule, the sums of money mentioned in the question can be expressed decimally; the answer, if obtained by the decimal method as far as the third place (301), will seldom be so much as one farthing wrong. It will be found very advantageous to use decimals throughout this Rule.

SIMPLE INTEREST.

372. In Simple Interest we have Principal, Rate per cent., Interest, and Time, to consider, and any three of these being given we are required to find the fourth.

The following are examples of each form of question:—

373. *Example I.*—What is the interest on £272 at 3 per cent.?

Here the answer is to be interest.

$\therefore$ £3, which is the known interest on £100, is the third term, and $3 : x$ is the ratio to be solved.

And by Rule of Three $100 : 272 :: 3 : x$.

$$\therefore x = \frac{272 \times 3}{100} = £8{\cdot}16 = £8\ 3s.\ 2\tfrac{1}{4}.$$

RULE FOR SIMPLE INTEREST.

$$\begin{array}{rr} & 272 \\ & 3 \\ \hline £. & 8,16 \\ & 20 \\ \hline s. & 3,20 \\ & 12 \\ \hline d. & 2,40 \\ & 4 \\ \hline f. & 1,60 \end{array}$$

Ans. £8 3s. 2¼d.

Multiply the principal by the rate per cent. and divide by 100.

For it is evident that in every sum in Interest the statement takes the same form: £100 is always the first term, the principal is always the second term, and the rate of interest always the third. The division by 100 may always be effected *by cutting off two figures from the right hand.* The sum is therefore worked as in the margin.

Observation.—The answer obtained by the former method, £8 3s. 2¼d., is evidently as near the truth as this answer, and is much more conveniently worked.

374. *Example II.*—From what principal did £273 15s. arise in a year, at 4½ per cent.?

Here the answer is to be principal. But cent., or £100, is the principal named in the sum, and $100 : x$ is the ratio to be solved.

Because the proportion between interest and principal is direct, ∴ the answer will be greater than £100;

And the statement is £4 10s. : 273·75 :: 100 : x.

$$\therefore x = \frac{273{\cdot}75 \times 100}{4{\cdot}5} = £6083{\cdot}\dot{3} = £6083\ 6s.\ 8d.$$

375. *Example III.*—At what rate per cent. will £720 amount to £785 in a year.?

Here the answer required is interest on £100.

But because £720 amounts to £785, the difference between these two, or 785 − 720, or £65, is the interest on £720.

∴ the statement is $720 : 100 :: 65 : x$.

$$\text{And } x = \frac{65 \times 100}{720} = £9{\cdot}026 = £9\ 0s.\ 6\tfrac{1}{4}d.$$

376. Whenever the condition of time is introduced the question is one in Compound Proportion.

Example IV.—What is the interest on £572 10s., at 4 per cent., for 6 weeks?

Here interest is to be the answer. ∴ $4 : x$ is the ratio required.

Two circumstances affect this ratio, principal and time.

I. If £4 is the interest on £100, what is the interest on £572 10s.?

The statement is 100 : £572 10s. :: 4 : x.

II. If £4 be the interest for one year, or 52 weeks, what is the interest for 6 weeks?

The statement is $52 : 6 :: 4 : x$.

Combining these two statements—

$$\left.\begin{array}{r} 100 : 572{\cdot}5 \\ 52 : 6 \end{array}\right\} :: 4 : x$$

$$\therefore x = \frac{572{\cdot}5 \times 6 \times 4}{100 \times 52} = £2{\cdot}643 = £2\ 12s.\ 10\tfrac{1}{2}d.$$

377. *Observation.*—This sum might be done by finding the year's interest according to the last rule, and then establishing a simple proportion, thus:—

As one year *is to* the given time, *so is* a year's interest *to the* interest for the given time.

378. *Example V.*—What is the principal from which £270 arises as interest in $4\frac{1}{2}$ years, at $6\frac{1}{4}$ per cent.?

Here the answer is to be principal, and $100 : x$ is the ratio to be solved.

The two questions to be considered are,—

I. If £600 is the principal from which £6·25 arises, what is the principal from which £270 will arise? This is a case of Direct Proportion, and the answer required will be greater.

$\therefore$ the statement is $6{\cdot}25 : 270 :: 100 : x$.

II. If £100 is the principal from which a certain interest arises in 1 year, what is the principal from which the same interest would arise in $4\frac{1}{2}$ years? This is a case of Inverse Proportion, and the answer required is to be less than £100.

$\therefore$ the statement is $4{\cdot}5 : 1 :: 100 : x$.

Combining these two statements—

$$\left.\begin{array}{r} 6{\cdot}25 : 270 \\ 4{\cdot}5 : 1 \end{array}\right\} :: 100 : x$$

$$\text{And } x = \frac{270 \times 100}{6{\cdot}25 \times 4{\cdot}5} = £960.$$

379. *Example VI.*—At what rate per cent would £1720 amount to £1865 in 5 years?

Here the answer required is interest, and £1865 − 1720 = 145 is the interest on the given sum, and $145 : x$ is the ratio to be determined.

I. If £145 be the interest on £1720 what is the interest on £100? The proportion is direct, and the answer required is less.

$\therefore$ *the statement* is $1720 : 100 :: 145 : x$.

II. If £145 be the interest for 5 years, what is the interest for 1?

This proportion is also direct, and the required answer is to be less. $\therefore$ the statement is $5 : 1 :: 145 : x$.

Combining these two statements—

$$\left.\begin{array}{r} 1720 : 100 \\ 5 : 1 \end{array}\right\} :: 145 : x$$

$$\text{And } x = \frac{100 \times 145}{1720 \times 5} = £1{\cdot}686 = £1\ 13s.\ 8\tfrac{3}{4}d.$$

380. The rules here illustrated may be compendiously stated thus:—

There are four items, viz., Principal, Interest, Rate per cent., and Time, any three of which being given, the fourth may be found.

Let P. = Principal, I = Interest, R = Rate per cent., and T = Time.

I. When Principal, Rate, and Time are given, to find the Interest—

$$I. = \frac{P.\ R.\ T.}{100}$$

II. When Interest, Rate, and Time are given, to find the Principal—

$$P. = \frac{100 \times I.}{R.\ T.}$$

III. When Principal, Interest, and Time are given, to find the Rate—

$$R. = \frac{100 \times I.}{P.\ T.}$$

IV. When Principal, Interest, and Rate are given, to find the Time—

$$T. = \frac{100 \times I.}{P.\ R.}$$

Exercise CVI.

1. Find the interest on £357 12s. at 5 per cent. for 1 year.
2. Find the interest on £4098 at $4\frac{1}{2}$ per cent. for $1\frac{1}{2}$ years.
3. Find the interest on £729 16s. at $3\frac{1}{4}$ per cent. for 2 years.
4. Find the interest on £874 13s. at $2\frac{1}{2}$ per cent. for 4 years.
5. Find the interest on £3096 10s. at 3 per cent. for $4\frac{1}{4}$ years.
6. Find the interest on £895 at $2\frac{1}{4}$ per cent. for 7 yrs. 2 months.
7. Find the interest on £3728 at $1\frac{1}{2}$ per cent. for 3 years 7 weeks.
8. Find the interest on £8547 at 2·75 per cent. for 8 years.
9. Find the interest on £3276 at $6\frac{1}{8}$ per cent. for 5 years.

10. Find interest on £8097 at 7⅓ per cent. for 7 weeks.

11. Find interest on £1813 19s. at 3¼ per cent. for 9 months.

12. Find interest on £5208 16s. at 7½ per cent. for 11 wks. 3 days.

13. Find interest on £7109 18s. at 3¼ per cent. for 2 yrs. 3 mths.

14. Find interest on £8297 13s. 6d. at 5½ per cent. for 17 months.

15. Find interest on £8634 15s. at 7¼ per cent. for 173 days.

16. Find interest on £718 10s. at 3⅓ per cent. for 2 years 94 days.

17. Find interest on £8274 12s. at 6¼ per cent. for 7 years 8 wks.

18. Find interest on £3274 at 3⅛ per cent for 21 years.

19. Find interest on £8067 15s. 3d. at 7⅛ per cent. for 6 yrs. 7 wks.

20. In what time will £723 15s. amount to £1280 at 3 per cent.?

21. In what time will £1072 16s. amount to £2000 at 5¼ per cent.?

22. In what time will £863 10s. amount to £1073 at 7¼ per cent.?

23. In what time will £427 amount to £500 at 2¼ per cent.?

24. At what rate per cent. will £832 amount to £1000 in 5 years?

25. At what rate per cent. will £79 amount to £100 in 7½ years?

26. At what rate per cent. will £1016 amount to £1250 in 5 years 5 months?

27. At what rate per cent. will £729 amount to £850 in 6¼ years?

28. From what principal will £86 accrue as interest in 2 years at 5 per cent.?

29. From what principal will £274 accrue as interest in 4 months at 3 per cent.?

30. From what principal will £18 accrue as interest in 7 weeks at 6 per cent.?

31. From what principal will £259 10s. accrue as interest in 3½ years at 4¼ per cent.?

32. How long will it be before £374 put out at interest at 4 per cent. will realize a profit of £100?

33. What sum must I invest at 3¾ per cent. so as to secure an annual income of £250?

34. If after lending £1560 for 4 years I receive the sum of £1870 in repayment, at what rate per cent. has the money been invested?

35. What time must elapse between the time of placing £28 in the Savings Bank and of taking out £43 10s., supposing interest is at 3 per cent.?

36. If I invest £1673 10s. at £3 15s. 6d. per cent. for 18 years *3 months*, to what sum will it amount?

COMPOUND INTEREST.

381. When the interest of money is added to the principal at certain periods, and afterwards interest is calculated on this amount, the money is said to be put to Compound Interest.

Every sum in Compound Interest consists of a series of sums in Simple Interest, a separate calculation being required for each of the periods mentioned in the sum.

Example.—What is the compound interest on £628 at 5 per cent. for 3 years?

By Rule (373) $\frac{628 \times 5}{100} = £31{\cdot}4 = £31\ 8s.$ = interest for the 1st year.

$£628 + £31{\cdot}4 = £659\ 8s.$ = principal at the beginning of the 2nd year.

$\frac{659{\cdot}4 \times 5}{100} = £32{\cdot}97 = £32\ 19s.\ 5d.$ = interest for the 2nd year.

$£659{\cdot}4 + £32{\cdot}97 = £692{\cdot}37 = £692\ 7s.\ 5d.$ = principal at beginning of 3rd year.

$\frac{692{\cdot}37 \times 5}{100} = £34{\cdot}6185 = £34\ 12s.\ 4\frac{1}{4}d.$ = interest for 3rd year.

$£692{\cdot}37 + 34{\cdot}618 = 726{\cdot}988 = £726\ 19s.\ 9\frac{1}{4}d.$ = amount at end of 3rd year.

Hence £628 has become £726 19s. 9¼d. in the three years, and the total interest which has accumulated on it is

£726 19s. 9¼d. − £628, or £98 19s. 9¼d.

382. A somewhat readier method of obtaining the result is that of finding the compound interest on £100 for the given time, and then solving the original question by a Proportion sum.

Thus £5 = interest on the first year.

∴ £105 = principal at beginning of 2nd year.

$\frac{105 \times 5}{100} = £5{\cdot}25 = £5\ 5s.$ = interest on 2nd year.

∴ $105{\cdot} + 5{\cdot}25 - £110{\cdot}25 = £110\ 5s.$ = principal at beginning of 3rd year.

But $\frac{110{\cdot}25 \times 5}{100} = £5{\cdot}5125 = £5\ 10s.\ 3d.$ = interest on 3rd year.

And $£110{\cdot}25 + £5{\cdot}5125 = £115{\cdot}7625 = £115\ 15s.\ 3d.$ = principal at end of 3rd year.

∴ £100 has accumulated to £115 15s. 3d. in three years, and £15 15s. 3d. is the compound interest on £100 for the given time.

We have now the following proportion:—

If £15 15s. 3d. be the compound interest on £100, what will be the compound interest on £628 for the same time and at the same rate?

The statement is 100 : 628 :: £15 15s. 3d. : x.

$$\text{And } x = \frac{15 \cdot 762 \times 628}{100} = £98 \cdot 988 = £98\ 19s.\ 9\tfrac{1}{2}d.$$

RULE FOR COMPOUND INTEREST.

383. Find the simple interest for each of the periods mentioned in the sum, and add their results successively to the principal. Or,

Find the compound interest on £100 for the given time, and then solve the question by Proportion, as in the example.

EXERCISE CVII.

1. Find compound interest on £500 at 4 per cent. for 3 years.
2. Find compound interest on £720 for 3 years at 4½ per cent.
3. Find compound interest on £1150 for 2½ years at 5 per cent.
4. Find compound interest on £484 for 5 years at 3½ per cent.
5. Find compound interest on £1250 for 2 years at 6 per cent.
6. Find compound interest on £7084 for 3 years at 3½ per cent.
7. Find compound interest on £2257 for 2½ years at 2½ per cent.
8. Find compound interest on £1097 for 5 years at 3½ per cent.
9. Find compound interest on £2384 for 4 years at 4 per cent.
10. Find compound interest on £5063 for 3 years at 2½ per cent.
11. To what sum will £187 amount in 5 years at 3 per cent. compound interest?
12. What is the compound interest on £1200 at 6 per cent. for 2½ years, the interest being paid half-yearly?
13. To what sum will £150 amount in 3½ years at 4 per cent., the interest being paid quarterly?

DISCOUNT.

384. *Discount* is a deduction made from a debt which is paid before it is due.

Questions of this sort occur in this Rule:—

A debtor is bound to pay £450 at the expiration of a year and a half from this date; what ought he to pay *now* to clear himself if interest is $4\frac{1}{2}$ per cent.?

Observe here that we must not calculate interest on the £450, for that is not the principal, but the *amount* to which his present debt will have accumulated if he leaves it unpaid for a year and a half. Whatever he owes at this moment is the principal, and this with the interest upon it at $4\frac{1}{2}$ per cent., will amount to £450 in a year and a half. We have therefore to separate £450 into two parts, so that one shall be the interest upon the other at $4\frac{1}{2}$ per cent. for $1\frac{1}{2}$ years.

As in the second case of compound interest, it will be convenient to take £100 as a standard, and find what it would amount to at the same rate and time.

$$\text{Now } \frac{100 \times 4{\cdot}5 \times 1{\cdot}5}{100} = £6\ 15\text{s.} = \text{interest on } £100.$$

$\therefore$ £100 present debt would become £106 15s. a year and a half hence at $4\frac{1}{2}$ per cent.

We have now the following proportion:—

If £100 will become £106 15s. in a certain time and at a certain rate, what sum will amount to £450 at the same rate?

The statement is $106{\cdot}75 : 450 :: 100 : x$.

$$\text{And } x = \frac{450 \times 100}{106{\cdot}75} = £421{\cdot}545 = £421\ 10\text{s. } 11\text{d.}$$

385. This answer gives the Present Value, and the difference between this and £450 is called the Discount. Or the discount itself might be found by the following proportion:—

If £6 15s. is the discount which should be deducted from £106 15s., what is the discount to be deducted from £450?

The statement is $£106\ 15\text{s.} : 450 :: £6\ 15\text{s.} : x$.

$$\text{And } x = \frac{450 \times 6{\cdot}75}{106{\cdot}75} = £28{\cdot}455 = £28\ 9\text{s. } 1\tfrac{1}{4}\text{d.}$$

386. The difference between the calculation of Interest and Discount is this, that in the one the principal is generally given and we are required to find the interest which has to be added to it; but in the other a sum of money is mentioned which includes both principal and interest, and we are required to separate it into those two parts. This latter problem is always to be solved by choosing a sum of money, finding what it would amount to under exactly the same circumstances as the unknown principal, and thus establishing a proportion. For convenience we generally select £100 for this purpose, but any other sum of money would enable us to obtain the same result.

387. *Example.*—What sum of money is that which, after lying in the Bank 7 years, when money is at 3 per cent., will amount to £5630?

We will choose £50 as the basis of the proportion:—

$\frac{50 \times 7 \times 3}{100} = £10{\cdot}5 = £10\ 10s. =$ the interest on £50 for the given time.

$\therefore$ £50 would become £60 10s. under the same circumstances.

If a principal of £50 would become £60 10s. in 7 years at 3 per cent., what principal is that which will become £5630 at the same rate and time?

The answer required is principal, and $50 : x$ is the ratio to be solved.

$$£60\ 10s. : 5630 :: 50 : x.$$

$$\text{And } x = \frac{5630 \times 50}{60{\cdot}5} = £4652{\cdot}892 = £4652\ 17s.\ 10\tfrac{1}{4}d.$$

Of course this answer is the same as if 100 had been taken, for then the statement would have been—

$$£121 : 5630 :: 100 : x.$$

$$\text{And } x = \frac{5630 \times 100}{121} = £4652{\cdot}892 = £4652\ 17s.\ 10\tfrac{1}{4}d.$$

388. Bills and Promissory Notes are written engagements on the part of a debtor to pay a certain amount on some future day. A *Bill* is drawn by the person to whom the money is due, and accepted or signed by the debtor. By accepting the bill he becomes legally liable for the amount when the time expires. A Promissory Note is simply signed by the party promising to pay. Three days *are* allowed by law beyond the date specified in the document; thus,

a Bill or Promissory Note made payable on the 3rd of September is not legally presentable until the 6th, unless this day should be Sunday, in which case it must be presented on the preceding day.*

Suppose A holds a bill by which B is bound to pay £500 this day three months, and A wishes to realize his money immediately. It is clear that if he parts with his claim or sells the bill, he ought to receive in lieu of it such a sum of money as, if put out at interest for three months, would amount to £500 by the end of that time. Many persons, as bankers and bill-discounters, are ready to negotiate in such matters and to buy bills before they are due, provided that the credit of the acceptor is good and he is considered likely to pay when the bill falls due. But, in fact, the true discount is never calculated on bills, but the *interest* on the sum for which the bill is drawn is deducted from the whole sum, and is called the discount. The discounter of the bill evidently derives a small advantage from this arrangement. For if 4 per cent. is the rate of interest, and he pays A £495 for the bill, deducting £5, or three months' interest on £500 at 4 per cent., the discounter really receives rather more than 4 per cent. for the money which he advances, for £495 put out at interest would *not* amount to £500 in three months.

Tradesmen often make an allowance to such of their customers as choose to pay ready money. This allowance is called Discount, but is really calculated as Interest, the per centage being found on the whole sum at a certain rate and deducted from it.

* The following is a form of Bill:—

To Mr. C. D., Liverpool. Gresham-street, London, Sept. 8, 1854.

Six months after date pay to me or to my order the sum of one hundred and fifty pounds, value received.

£150 0s. 0d. A. B.

C. D. *accepts* the bill by writing his name either under that of A. B., the *drawer*, or across the bill.

A Promissory Note has the same force, and subjects the debtor to the same obligation. Its form is generally as follows:—

Liverpool, Sept. 8, 1854.

Six months after date I promise to pay to Mr. A. B. or his order the sum of one hundred and fifty pounds. C. D.

No other signature is required here than that of C. D., the debtor, but such a note is usually endorsed by the creditor.

RULE FOR DISCOUNT AND PRESENT VALUE.

389. Take £100, find what it would amount to at the same rate and time, and state the proportion as follows:—

As £100 + *its interest for the given time* is to *the given amount*, so is £100 to the present value.

General Formula.—If A = amount, and I = interest on £100—

$$\text{Then Present Value} = \frac{100\ A}{100 + I}$$

Or if the discount be required—

As £100 + *its interest* is to *the given amount*, so is *the interest on* £100 to the answer.

$$\textit{General Formula.}\text{—}\frac{A \times I \text{ on } £100}{100 + I} = \text{Discount.}$$

In the following Exercise the expression, *true discount*, refers to the calculation made in this way, while *ordinary discount* means the deduction actually made in business.

Exercise CVIII.

1. What sum will discharge a debt of £720, due a year and a half hence, at 4 per cent.?

2. What sum put out at interest will amount to £1310 14s. in six years at $4\frac{3}{4}$ per cent.?

3. If a legacy of £1200, less 5 per cent. duty, is to be paid to a person whose age is 17 when he becomes 24 years old, what sum paid to him *now* would be equivalent to it?

4. What is the difference between the ordinary bankers' discount on a bill of £470 due six months hence, at 5 per cent., and the true discount?

5. A tradesman accustomed to give nine months' credit is offered ready money for an account of £58 14s. 9d.; what is the ordinary discount at 5 per cent.?

6. Find the true discount on £107 5s., payable at the end of six months, at $3\frac{1}{2}$ per cent.

7. What is the exact present value of a debt of £572, due eight *months* hence, at £3·75 per cent. simple interest?

8. How much money must I invest at 7 per cent. in order that at the end of four and a half years I may be worth £5000?

9. Find the difference between the true and the ordinary discounts in the case of the following bills:—

(a). £450 drawn on March 1, payable on June 1, at 4 per cent.
£1000 drawn on June 18, payable on August 18, at 5 per cent.
£1728 drawn on Sept. 27, payable on Dec. 31, at $3\frac{1}{2}$ per cent.

(b). £2347 drawn on Aug. 26, payable on Jan. 15, at $2\frac{1}{4}$ per cent.
£6274 drawn on Feb. 13, payable on June 16, at $3\frac{3}{4}$ per cent.
£240 drawn on April 3, payable on July 1, at $2\frac{1}{4}$ per cent.

(c). £27 drawn on Jan. 6, payable on April 28, at $3\frac{1}{4}$ per cent.
£48 drawn on Dec. 5, payable on Jan. 8, at 5 per cent.
£623 drawn on Dec. 28, payable on April 3, at 4 per cent.

STOCKS AND SHARES.

390. In the reign of William III. the Government of England laid the foundation of our present National Debt by borrowing money from private persons, which was employed for the necessities of the State.* In order to induce persons possessing capital to dispose of their money in this way, interest was offered at a certain fixed rate per annum. At different periods since that time foreign wars and other national emergencies have made it necessary for the Government to borrow more money of individuals; and the total sum which has been thus lent to the State and expended by it, is nearly £800,000,000. Every person whose money has been thus disposed of receives interest regularly every half-year, at the rate of 3 or $3\frac{1}{4}$ per cent., and also holds an acknowledgment from the Government, of the debt. These acknowledgments may be transferred from one name to another, and may be bought by any person who chooses. Like everything else which is marketable, they fluctuate in value, becoming dear when there is a general disposition on the part of capitalists to buy them, and becoming cheap when many of the

* Up to this period money had often been borrowed by Government, but generally on the security of some special tax. A king in difficulty would mortgage the "tonnage and poundage," for example, and if it failed to produce the necessary sum, the lenders lost money by the transaction and were never repaid. But it was in this reign that the national credit was for the first time pledged to the lenders, and that the *permanent* debt was established.

holders are anxious to sell. Now when the prospects of the country are unusually good, and there is a greater probability than usual that the debt will be reduced; or, when money is so abundant that persons possessing it cannot readily find other investments, and so are glad to get the 3 or 3¼ per cent. which Government allows, buyers of stock are numerous, and prices advance. But the prospects of war, the chances of a bad harvest, or such a scarcity of money as makes people anxious to employ their capital in other ways, will lessen the number of buyers, and cause the stock to become cheap. The extent of speculation, and many other circumstances, also affect the value of the funds, so that it is seldom that prices remain perfectly stationary even for two consecutive days. Thus: a £100 share, or £100 stock as it is called, rose in value in 1752 to £106 7s. 6d., the maximum ever attained, and in September, 1797, could be bought for £47 12s. 6d., which is the lowest price at which they were ever quoted.

When the market price of £100 stock is £100, the funds are said to be *at par*. They are usually below par; and in ordinary times prices fluctuate between 91 and 98.

The person in whose name the stock stands at the end of each half year, receives the dividends, and therefore a purchaser has the benefit of the interest on the stock he buys, from the last day of payment to the day of transfer. Hence, all other things being equal, the approach of the dividend day causes the price of stock to advance, and after the dividend has been paid prices experience a decline.

Dividends on different stocks are paid at different times in the year, so that it is easy for a stock-holder to invest his money so as to receive his income quarterly.

391. It is evident that if a person takes a share in this debt, or purchases Stock as it is called, when funds are low, he is credited with a larger sum than he actually invests; and as he receives £3 or £3 5s. per cent. on this nominal value, the interest actually accruing on his capital is greater than the nominal interest. The price of the funds is therefore a good measure of the rate of interest, and the general abundance of money,—for Interest is high when Funds are low, and *vice versâ*. If funds rise in price, and a man sells his shares at a higher value than that at which he bought them, he may obtain profit by the transaction just as in ordinary trade. Many

persons make a business of the purchase and sale of stock with a view to profit in this way, by speculating on the probabilities of a rise or fall in the funds.*

392. Sometimes foreign governments desire to raise a loan, and invite all the capitalists of Europe to take shares in it. Bonds, or engagements to pay interest until the money is returned, are issued and sold in the money markets. If the credit of the government wishing to borrow be generally considered good, and the interest offered be high, many persons will become eager to take up portions of the loan; the prices of the stock advance, and the actual interest on the investment is proportionately diminished. It is evident that the market price of foreign bonds or securities is affected by the same circumstances as that of the English funds.

393. Shares in railway, canal, or other public companies are also bought and sold in the share market, and are regulated by the same laws. For example, suppose a company is started requiring a capital of £1,000,000, and this sum is distributed in the form of 100,000 shares of £10 each. If the prospects of the company are generally considered very good, and it is likely to realize a handsome profit on its capital, many persons will desire to become shareholders, and the shares will rise in value. Perhaps a £10 share will be sold for £12: in this case the value of the company's shares is said to be £2 *premium;* but if, on the other hand, persons are unwilling to join in the undertaking, the shares will probably be sold at a *discount, i. e.,* at less than their nominal value.

394. Transfers of invested capital from one form of stock to another are very common. Whenever this is done the nominal value of the given sum is in inverse proportion to the market price of the stock. Thus: if I have a sum of money invested in Russian

* For example: A engages to sell B £1000 stock for £900 on a certain fixed day, perhaps 3 months hence. A possesses in fact no such stock, but he has reason to believe that by the time mentioned the price of stock will be below 90, and in that case he will gain a profit,—for should funds be at 88, he will be able to buy £1000 stock for £880, and so to gain £20 by the transaction. B however engages to buy in the hope that funds will be at a higher price than 90, and thus to gain a profit by purchasing stock at that price, and selling it at the market rate. In the slang of the Stock Exchange, A would be called a *Bear*, and B, who speculates on the chance of a rise, is called a *Bull*. Such bargains, though very common, are a sort of gambling, and are not recognized by law.

securities, which are at 57, and I transfer it to the English funds at 88½, the sum standing in my name in the latter will be as many times less than in the former as 57 is less than 88½. For if I were to sell the Russian stock, I should only realize £57 for every nominal £100, and with this sum I could not purchase nearly so large an amount of stock at 88½.

395. The persons who effect the necessary sales, and whose business it is to transfer stock and shares from one name to another, are called *Brokers*, and the sum they receive for their trouble is called *Brokerage*. The commission thus paid to Stock-brokers is always ⅛ per cent., or 2s. 6d. on every £100 stock in the funds, but is higher for other stock. It must always be added to the cost of purchasing stock, and subtracted from the receipts of one selling stock.

It is evident that the principles of proportion apply to all calculations of this kind. The following are examples of the sort of questions which occur in these rules:—

396. *Example I.*—How much stock in the 4 per cents. can be purchased for £1240 when the price is 89¾?

Here is a case of Inverse Proportion, for the lower the market value of the stock, the more can be purchased for a given sum.

The answer required is to be as many times greater than £1240 as 89¾ is less than 100.

$\therefore$ the statement is $89\frac{3}{4} : 100 :: 1240 : x$.

$$\text{And } x = \frac{1240 \times 100}{89\frac{3}{4}} \quad 1381{\cdot}615 = £1381\ 12s.\ 3\tfrac{3}{4}d.$$

397. *Example II.*—What will it cost to purchase £1050 stock when funds are at 93¼; brokerage at ⅛ per cent.?

Here the expense of brokerage must be added to the value of the stock, as it will have to be paid by the purchaser.

The case is one of Direct Proportion, for the lower the value of £100, the less money will be required to purchase any given nominal amount.

$\therefore$ the statement is $100 : 93{\cdot}375 :: 1050 : x$.

$$\text{And } x = \frac{1050 \times 93{\cdot}375}{100} = 980{\cdot}4375 = £980\ 8s.\ 9d.$$

398. *Example III.*—What sum will stand in my name in the 3 *per cents.* at 81½ if I transfer £1450 from Russian stock at 54½?

Here is a case of Inverse Proportion, because the lower the price of the stock the greater the nominal value of a given sum.

$\therefore$ the statement is $81{\cdot}25 : 54{\cdot}5 :: 1450 : x$.

$$\therefore x = \frac{1450 \times 54{\cdot}5}{81{\cdot}25} = 972{\cdot}615 = £972\ 12s.\ 3\tfrac{3}{4}d.$$

Exercise CIX.

1. What sum invested in the 3 per cents. when they are at $98\frac{1}{4}$ will have the nominal value of £7268?

2. What annual income should I derive from an investment of £3500 in the $3\frac{1}{4}$ per cents. if I buy the stock at $91\frac{1}{2}$?

3. What is the price of stock when I can buy £3158 for £2754 10s.?

4. If I lay out £840 in the purchase of stock at $79\frac{1}{2}$, and sell out at $85\frac{3}{8}$, what do I gain?

5. If I purchase 120 Bank Annuities at $90\frac{3}{8}$, at what price must I sell them so as to gain £150?

6. What is the difference between the annual income arising from the investment of £2150 in the $3\frac{1}{4}$ per cents. at $87\frac{1}{8}$, and that from the same sum in the 3 per cents. at $86\frac{1}{2}$?

7. A person invests £1248 in the 3 per cents. at $95\frac{1}{2}$; what will be his net half-yearly dividend after deducting 7d. in the pound per annum for income tax?

8. What would be realized by selling £8296 10s. stock, at $88\frac{5}{8}$?

9. If funds are at $82\frac{1}{8}$, what must be given for £1250 stock?

10. How much should I gain by purchasing £700 stock at $97\frac{3}{8}$, and selling out at $101\frac{1}{4}$?

11. When the 3 per cents. are at $89\frac{1}{4}$, at what rate may the same quantity of stock be purchased in the $3\frac{1}{2}$ per cents. with equal advantage?

12. A person invests £3000 in the 3 per cents. when they are at $76\frac{1}{8}$, what will be his annual income?

13. How much stock at $91\frac{1}{8}$ can be bought for £768, $\frac{1}{8}$ per cent. being charged for commission?

14. Suppose I lay out £1270 in the 3 per cents. at $92\frac{1}{2}$, and sell out after allowing the interest to accumulate for two years, and find myself the richer by £147 10s.; at what price do I sell out?

Miscellaneous Applications of the term Per Cent.

399. One hundred is employed as the standard of proportion in many other cases besides Interest, *e.g.*,

Profit and Loss. Tradesmen measure their gains and losses on the whole of the capital they invest, by a certain sum *per cent.*, or on every £100 of their outlay.

400. *Commission.* Travellers, agents, collectors, and others, who are employed by a firm or a company, either to receive money or to extend business, are usually paid at the rate of so much *per cent.* on every £100 which passes through their hands.

401. *Insurance* against casualties, such as premature death, fire, or the loss of a ship at sea, is always calculated at a certain fixed rate per £100 on the sum insured. Thus: if a man at a certain age (say 40) desires to insure his life, a calculation is made founded on the average rate of mortality among persons of that age, and the insurer is charged with such an annual sum, as if he lives to the average age, will repay the insurance company and give them a reasonable profit. So also if a house or warehouse has to be insured against fire, the probability of the house taking fire is calculated from the ascertained number of fires in proportion to the total number of houses and warehouses; and this is determined at the rate of so much per £100. Insurances of life as well as of houses, warehouses, &c., are usually effected by a company, as such dealings are necessarily on a large scale and involve too large a risk for private persons to incur. But the insurance of vessels at sea is usually undertaken by private persons who are called underwriters. It is thus effected. A is the owner of a ship whose cargo is worth £5000; he desires to indemnify himself against the loss of the ship at sea; he therefore proposes to various underwriters to take shares in the insurance: one will perhaps take £500 worth of the risk, and another will engage himself to the extent of £1000; each will receive a sum by way of insurance proportioned to the amount of risk he undertakes; and in the event of the loss of the ship, each will be bound to pay to the owners the sum for which he became responsible.

402. *Observation.*—It is to be noticed here that in the case of human life, of fire, of shipwreck, or of loss of life by railway, the amount of money paid as insurance is regulated by the *probability*

of some particular event. This must in every case be determined by past experience. Suppose, for example, it is proved on referring to the history of past years, that one ship in 200 of all which go from London to the West Indies founders or is lost; then the underwriter is bound to assume that the chance of any particular ship reaching its destination safely is as 200 to 1. He therefore would be safe if he insured every ship going to the West Indies at $\frac{1}{200}$ part of the value of the cargo, because the doctrine of chances would justify him in expecting that 200 shipowners would pay him insurance for every *one* whom he would have to compensate for loss. By carefully keeping registers of casualties, the amount of risk incurred by life or property, either on the sea or in travelling, may be pretty accurately measured, and all insurance tables are founded on such registers. In all these the proportion is reckoned at so much per cent.

403. *Statistical tables* respecting population, employments, education, &c., are always constructed on the basis of 100, and are calculated at *per cent.* Thus: if it be found that in a country containing $2\frac{1}{2}$ millions of inhabitants, a certain number, say 400,000, are employed in agriculture, it is usual to express this fact by stating that a certain number per cent., or out of every 100 of the population, are so employed.

The following are examples of the use of this important formula; and it is evident that, as in every case 100 is only assumed for convenience as the standard of proportion, the principles of the Rule of Three apply to them all.

404. *Example I.*—A man invests £2340 and makes £2587 of it; what does he gain per cent.?

Here his total gain is £2587 − 2340 = £247.

But the gain on £100 will be less than on £2340.

∴ the statement is $2340 : 100 :: 247 : x$.

$$\text{And } x = \frac{247 \times 100}{2340} = 10\cdot\dot{5} = £10\ 11s.\ 1\tfrac{1}{3}d.$$

405. *Example II.*—An agent who sells an estate worth £12346, receives a commission of 3s. 6d. per cent; how much does he get by the transaction?

Here the statement is $12346 : 100 :: \cdot175 : x$.

$$\text{And } x = \frac{100 \times 1\cdot75}{12\cdot346} \text{ the answer being in pounds.}$$

$$\text{Or } x = \frac{100 \times 3\cdot5}{12\cdot346} = \text{the answer in shillings.}$$

406. *Example III.*—In a country containing a population of 14,500,000, 3,270,000 derive their subsistence from agriculture, 4,820,000 from trade, 2,748,000 from manufactures, 1,125,000 from professions, 975,000 are persons of independent means, and the rest are paupers; what is the per centage of population in each class.

Here the proportions in all cases are direct, and the statements are—

14500000 : 100 :: 3270000 : per centage engaged in agriculture.
14500000 : 100 :: 4820000 : per centage engaged in trade.
14500000 : 100 :: 2748000 : per centage engaged in manufactures.
14500000 : 100 :: 1125000 : per centage engaged in professions.
14500000 : 100 :: 975000 : per centage of independent persons.
14500000 : 100 :: 1562000 : per centage of paupers.

It will be found that all these answers added together will make 100, and will represent the proportions in which an average 100 of the population are distributed among the various classes of society.

EXERCISE CX.

1. In a school of 250 children 44 per cent. are learning geography, 36 per cent. are learning grammar, 12 per cent. cannot read, and only 4 per cent. have advanced as far as algebra. What are the actual numbers of each?

2. An agent who is paid $2\frac{3}{4}$ per cent. on all the money he collects, receives £57 as commission; how much has he collected?

3. There are two schools, one containing 650, and the other 340 children: 5 per cent. of the former are generally absent, and 7·5 of the latter; what is the average attendance in each?

4. Air is composed of 3 gases, 75·55 per cent. being nitrogen, 23·32 oxygen, and 1·13 carbonic acid. In a chamber containing 3274 cubic feet, how much is there of each gas?

5. In the year 1852 the total number of persons killed on all the railways of the United Kingdom was 216. Of these, 32 were passengers, 120 were railway servants, and the rest were trespassers. Find the per centage of each class.

6. In the same time the total number of journies made on the railways was 89,145,729. What was the per centage of passengers who lost their lives?

7. In 1831 the population of Great Britain and the islands in the British seas was 16,364,893; in 1841 it was 18,658,372; in

1851 it was 20,936,468: what was the ratio of increase per cent. during each decennial interval?

8. In 1841 the population of Ireland was 8,175,124; in 1851 only 6,515,794: what was the decrease per cent?

9. Of 138918 persons convicted of various offences during 5 years, 30·66 per cent. were unable to read and write; 58·89 per cent. could do both fairly; and the rest had received a decent education: what were the numbers in each class?

10. The number of children belonging to all the public schools in England at the last census was 1,407,569; the number belonging to all the private schools was 700,904. Of the former, only 1,115,237 were actually present on the day of the census; of the latter 639,739 were in attendance: compare the per centage of attendance in public schools with that in private.

11. At the same time the population of England and Wales was 17,927,609; what per centage of the people was under tuition?

12. What is the insurance on £7285 at £2 7s. 6d. per cent.?

13. If the rate of insurance be £1 6s. 4d. per cent., for how much is a person insured who pays an annual premium of £29 10s.?

14. If a warehouse contains goods worth £17230, and is only insured for 86·3 per cent. of its value, what sum would be lost in case of its destruction by fire?

15. By selling 26 yards at 3s. 4½. per yard, a draper gains 6s. 6d. What was the prime cost per yard, and what is the gain per cent.?

16. If a dishonest fruit vender uses a weight of 14·76 oz. for 1 lb. and professes to sell his goods at the cost price, what does he gain per cent.?

17. If a grocer mixes 17 lbs. of tea worth 4s. with 25 lbs. worth 4s. 8d. and sells the whole at 5s. 4d. per lb., what is his total gain, and his profit per cent.?

18. By selling goods for £483 10s. a tradesman gets a profit of 7½ per cent.; what did he give for them?

19. If 1 cwt. 3 qrs. are purchased for £27 10s., what should be the retail price per lb. to give a profit of 5½ per cent.?

20. Suppose a man insures his life for £1000 at the yearly rate of £60, and dies immediately after he has paid his fifth annual premium; how much does the company lose by the transaction, reckoning money worth £5 per cent. compound interest?

Questions on Ratio and Proportion.

What is meant by *absolute* and *relative* magnitude? What is the usual standard of comparison? Distinguish Integral from Fractional, and both from Proportional Arithmetic. In how many ways may two magnitudes be compared? Which is the more usual of the two? Give a name to both. Define Ratio. Show how the definition given in the common translation of Euclid is open to objection? What sign is employed to indicate ratio? Define antecedent and consequent.

Why should the words "of like kinds" form part of the definition? If ratio be itself a magnitude is it abstract or concrete? Why? How far do the truths concerning fractions apply to ratio? Give examples. What is the effect on a ratio of diminishing its consequent? of increasing it? of diminishing its antecedent? of increasing it? of increasing both? of diminishing both? Give numerical illustrations.

Define Proportion. Make three proportions, and explain how they illustrate your definition. How far do the words "of like kinds" apply in this case? Why? By what test can you determine whether four numbers are in proportion? Will the test apply in the case of incommensurable magnitudes? Why? Prove in three ways that if $w : x :: y : z$, then $wz = xy$. What practical use is made of this truth in arithmetic?

How is any one term of a proportion to be determined when the other three are given? Suppose the product of any two numbers equals the product of any other two, what inference can you deduce from the fact? State the principle, prove it, and give two examples.

When is a ratio expressed in its lowest terms, and why? Give an example. What is Continued Proportion? What name is often applied to it? Give examples. How many other conclusions can you deduce from the fact that four numbers are in proportion? Give a reason in each case. If I square every number in a proportion how do I know that the resulting numbers are in proportion? Of what general truth is this a special example?

What is the purpose of the Rule of Three? Why is it so called? In the solution of a sum in Simple Proportion what two things have to be done? How are we to do the first? the second? What is Direct Proportion? Give other examples not mentioned in the text. Define Inverse Proportion. Give examples. How does a fraction illustrate the difference between Direct and Inverse Proportion? Take the two fractions $\frac{4}{11}$ and $\frac{4}{13}$ and show what principle explains their relation to each other.

How is a Rule of Three sum to be stated? Why? How is it to be solved after stating? Why? When I multiply the second and third terms together and divide by the first, what truth is taken for granted?

When and how are ratios compounded? Give examples. What operation in Fractions resembles the composition of ratios? On what principle does the composition of ratios depend? Show how the principle applies. Into what parts may every problem in Compound Proportion be resolved? Take the first four sums in Compound Proportion and resolve each of them into two distinct sums in Simple Proportion. Why should the two statements be combined?

Define interest, principal, rate per cent. Show how far interest illustrates Direct Proportion, and in what cases Inverse Proportion occurs. Give in symbolical form the rules for finding interest, principal, rate, or time. What is Compound Interest? Give an example, and the rule for working it. Why should we use the decimal method throughout this Rule? What is Discount, and how does it differ from Interest? To what extent is this Rule actually used in business?

What are Stocks? Explain why they fluctuate in value. Show how the price of Stocks determines the current rate of interest. What is Brokerage? Give an example. Give some other examples of the use of the term *per cent.*

State the following sums:—

Find the interest on £a at b per cent. for c months, and also the discount.

What will a articles cost at the rate of b shillings for n articles?

If a men dig b cubic feet in c days, how many will do e cubic feet in m days?

If stocks are sold at a, how much can I buy for £b?

For what sum can I buy £a stock when the price is b?

If a cwt. of goods are carried b miles for a certain sum, to what distance would c cwt. be carried?

Express in words the truths contained in the following formulæ:—

If $a : b :: c : d$ Then $a : c :: b : d$, and $c : a :: d : b$

Also $a \pm b : b :: c \pm d : d$, and $a \pm c : b \pm d :: a : b$

And $ma : mb :: c : d$, and $a : b :: \frac{c}{n} : \frac{d}{n}$

And $a^n : b^n :: c^n : d^n$, and $ad = cb$.

MISCELLANEOUS EXERCISES ON PROPORTION.

1. If 7 men can reap 6 acres in 12 hours, how many men will reap 15 acres in 14 hours?

2. What is the amount of £720 16s. in 3 years at 4 per cent. compound interest?

3. By how much does the compound interest on £527 for $4\frac{1}{2}$ years at $2\frac{1}{4}$ per cent. exceed the simple interest on the same sum for the same time?

4. If 800 soldiers consume 5 barrels of flour in 6 days, how many will consume 15 barrels in 2 days?

5. Find the discount on £125 8s. for 6 months at $3\frac{1}{4}$ per cent.

6. What is the difference between the interest on £135 7s. 6d. for 9 months at 4 per cent., and the discount on the same sum?

7. If the population of Great Britain be 18,526,830, and $5\frac{1}{4}$ per cent. possess the elective franchise, determine the number of electors; also state what per centage of the whole population of Great Britain is that of Scotland, which is 2,620,180?

8. If $17\frac{3}{4}$ ells, each containing 5 quarters, cost £6 17s., how much will 18 yards cost?

9. A bankrupt's stock was sold for £520 10s., at a loss of 17 per cent. on the cost price: had it been sold in the ordinary course of trade it would have realized a profit of 20 per cent; how much was it sold at below the trade price?

10. If the quartern loaf be sold for $7\frac{1}{2}$d. when wheat is 47s. per quarter, what should be its price when the price of wheat is 65s.?

11. A parcel of goods bought for £18 was sold 4 months afterwards for £25; what was the gain per cent. per annum?

12. If 8 oz. of bread are sold for 6d. when wheat is £15 a load, what should be the price of wheat when 12 oz. are sold for 4d.?

13. An annuity of £50 is put out to interest immediately after each payment; what will it amount to in 7 years, allowing 5 per cent. simple interest?

14. A person invests £1000 in 3 per cent. stock at $88\frac{3}{8}$ per cent.; what will be the amount of his half-yearly dividends?

15. What annual income will arise from two investments, the one of £1472 at 3 per cent., and the other of £2000 at $4\frac{1}{4}$ per cent., deducting income tax at 7d. in the pound?

16. What weight ought to be carried $25\frac{3}{4}$ miles for the same sum for which 3 cwt. are carried 40 miles?

17. What is the present value of £75, due 17 months hence, at 4 per cent.?

18. A man can reap $345\frac{5}{7}$ square yards in an hour; how long will 7 such labourers take to reap six acres?

19. A corn-factor buys 2 quarters of wheat at 39s. per quarter, and 7 bushels of a superior quality at 6s. per bushel; at what rate must he sell the mixture so as to gain £1 by the transaction?

20. At what rate per cent. will a person receive interest who invests his capital in the 3 per cents. when they are at 91?

21. A bequest of £468 is made to each of 3 persons: the first being a son of the testator pays a duty of 1 per cent.; the second being a brother pays 3 per cent.; and the third, who is not related, pays 10 per cent.: what sum does each receive?

22. A person transfers £1000 stock from the 4 per cents. at 90 to the 3 per cents. at £72; find how much of the latter stock he will hold, and the alteration made in his income?

23. What principal lent at simple interest on the 1st of January at $5\frac{1}{2}$ per cent. would amount to £1000 on the 29th of September in the same year?

24. A bankrupt owes £1537 3s. 4d. but can pay only £960 14s. 7d.; what will be the dividend? How much shall a creditor receive for a debt of £276 11s. 6d.?

25. At what price must an article be bought, that, being sold for £3 10s. 6d., 13 per cent. may be cleared?

26. Reduce 2746 American dollars 30 cents to British money, exchange being at 4s. $3\frac{1}{2}$d. British per dollar?

27. What sum must be put out at 4 per cent. compound interest to amount to £1000 in 5 years?

28. The divisible receipts of a railway company for one year are £437500, and there are 250000 shares at £21 each; what is the dividend on each share, and what is the rate per cent. of the company's profits?

29. What annual income will arise from the investment of £1800 in the $3\frac{1}{4}$ per cents. when they are at $87\frac{1}{8}$?

30. I borrow £130 on the 5th of March and pay back £132 10s. 6d. on the 18th of October; what rate per cent. per annum of interest have I paid?

31. A and B rent a field for £35 a year: A puts in 6 horses for the whole year; B, 5 horses for 11 months, and 3 more for 5 months: how much should each contribute towards the rent?

32. If 12 men build a wall 60 feet long, 4 thick, and 20 high in 24 days, working 12 hours a day, how many must be employed to build a wall 100 feet long, 3 thick, and 12 high in 18 days, working 8 hours a day?

33. A and B exchange goods: A gives 13 cwt. of hops, the retail price of which is 56s., but in barter he rates them at £3; B gives A 10 barrels of beer, the retail value of which is 1s. per gallon, but the value of which he raises in proportion to the increased value of the hops: how much must B give in money?

34. In what time will the sun move through 50·1 seconds when it describes 360 degrees in 365 days 6 hours 9 minutes 9·6 seconds, the motion being supposed uniform?

35. In what time will £645 15s. amount to £960 11s. $0\frac{3}{4}$d. at $4\frac{1}{8}$ per cent. per annum?

6. In what time will any sum of money double itself at $3\frac{1}{2}$ per .t. simple interest?

37. At what rate will any sum of money double itself in $4\frac{1}{2}$ years?

38. The total number of degrees in the five angles of a pentagon 540; find the angles of a pentagon which shall be to one another s 2, 3, 4, 5, and 6.

39. In discharging a debt of £200 due a year hence, allowing 5 per cent. simple interest, why ought I to pay more than £190, and how much more?

40. A tenant pays corn rent of 30 quarters of wheat and 12 of barley, Winchester measure; what is the value of his rent, wheat being at 60s. and barley at 64s. a quarter imperial measure, supposing a Winchester bushel to be to an imperial bushel as 32 to 33?

41. The mint price of gold is £3 17s. $10\frac{1}{2}$d. per ounce; what is the smallest number of exact ounces that can be coined into an exact number of sovereigns?

42. A tradesman marks his goods at two prices, one for ready money and the other for 6 months' credit; what fixed ratio ought the two prices to bear to one another, allowing 5 per cent. per annum simple interest? and what is the credit price of an article in his shop marked for ready money at £120 10s.?

43. Two companies of soldiers start at the same time to march 39 miles, but one by marching a quarter of a mile an hour faster than the other arrives there an hour sooner; required their rates of marching?

44. A man pays a tax of 10 per cent. on his income, but after it is paid he has £1250 per annum to spend; what is his annual income?

45. At 1s. $7\frac{1}{2}$d. per day, how long will a man be in saving £10?

46. Out of an allowance of £50 a year, how much will be saved after spending 2s. $6\frac{1}{2}$d. a day?

47. A man spends 19s. 6d. a day and saves 10 guineas every quarter; what is his income?

48. What sum of money invested at $4\frac{1}{2}$ per cent. will amount to £1000 in a year and a half?

49. At what rate per cent. is the profit which a stationer makes who sells a book at a reduction of 2d. in the shilling on the published price, and purchases it at 25 per cent. discount, the account running in the wholesale 12 months, in the retail 6 months?

INVOLUTION AND EVOLUTION.

407. The products formed by multiplying the same number into itself are called the *powers* of that number.

Thus the product of three sevens, or $7 \times 7 \times 7$, is called the third power of 7; the product of five sevens, or $7 \times 7 \times 7 \times 7 \times 7$, is called the fifth power of 7. These two results are written 7^3 and 7^5.

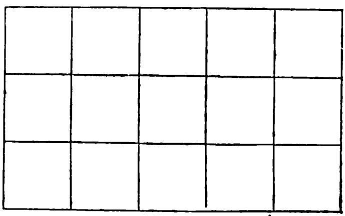
The figure which represents the power to which a number is raised is called the *Exponent;* thus, in 7^3 and 7^5 the 3 and 5 are exponents, the one showing that 7 is to be raised to the third, and the other that it is to be raised to the fifth power.

The second power of a number is called its *square*, and the third its *cube*.*

408. When we find the result of multiplying a number

* These terms are taken from geometry, and their use may be accounted for by the fact that when the manner of forming the second and third powers of numbers was first investigated it was with a view to the measurement of surfaces and solids. There are two simple propositions which will justify this connexion of arithmetic with geometry, and which are self-evident as soon as stated.

I. *If any two numbers represent the units of length in the adjacent sides of a rectangular parallelogram, their product will represent the units of surface in the parallelogram itself.*

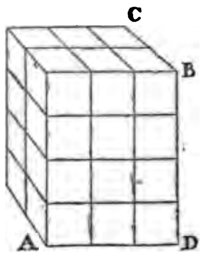


Thus: because one side contains 5 units of a certain length and the other side 3 units of the same length, the whole parallelogram contains 5×3, or 15 square surfaces, each having the unit of length for one side.

Corollary.—Whenever the two sides of the parallelogram are equal and the figure is a square, the number representing the area is the second power of the number representing the length of the side. Hence the second power is often called a *square.*

II. *If any three numbers represent the units of length of the three dimensions of a rectangular solid, the product of those three numbers represents the units of solidity in the body itself.*

by itself a certain number of times, we are said to *involve* that number to the given power, and the operation is called *Involution*. Thus $5 \times 5 \times 5 = 125$. The number 5 has here been *involved* to the third power.



Thus: because the height BD is represented by 4, the length AD by 3, and the width BC by 2, the whole mass contains $4 \times 3 \times 2$, or 24 regular solids, each having the given unit of length for each of its dimensions.

Corollary.—Whenever the solid is equal in all its dimensions and is therefore a cube, it is evident that the number representing the solidity is the third power of the number which represents the length of each dimension. Hence the third power of a number is often called a *cube*.

In old books several other terms, borrowed from mensuration, were employed in arithmetic. Thus in the 8th Book of Euclid the product of any two numbers is called a *plane* number, because it is a number which might represent the extent of a surface; and the product of any three numbers is called a *solid* number, because it might represent the magnitude of a rectangular mass. But these expressions have become obsolete, and as there is evidently nothing in geometry analogous to the fourth, fifth, or any higher powers of a number, the only geometrical terms employed in modern arithmetic are square and cube.

It follows, from these considerations, that any theorems in geometry which show the relative magnitude of rectangles and squares formed upon certain lines, will illustrate that part of arithmetic which relates to the formation of the products of numbers and their second powers. For example, the 2nd Book of Euclid investigates the conditions under which squares and rectangles are equal, and the relations of the lines and parts of lines on which they are constructed: to every proposition in that book therefore there is a corresponding proposition referring to the products of numbers. In like manner the truths of solid geometry will point to some analogous truths in relation to the products of numbers and their third powers. But it must be remembered that, although in these cases arithmetical and geometrical theorems illustrate one another, we must not attempt to prove the one by proving the other. The evidence of a geometrical proposition lies in certain axioms and elementary truths, which are founded on our idea of *space*; but the proof of all arithmetical propositions rests on axioms and notions of number only. Hence the propositions in the 2nd Book of Euclid cannot be proved numerically or algebraically, as some students suppose; they must be demonstrated as pure geometry, and the idea of number should be as far as possible excluded. In like manner there can be no geometrical proof of the rules for the involution of numbers; and although we shall find many propositions in this subject analogous to familiar theorems in geometry, their evidence depends on the principles of arithmetic alone.

409. But when a certain power of a number is given and we are required to find that number of which it is the power, we are said to *extract* or *evolve* the root, and the process is called *Evolution*. Thus to find that 125 is made up of $5 \times 5 \times 5$ is to evolve the 5, and 5 is called the third or cube *root* of 125.

Every number may be a *root*, because it may be involved or multiplied by itself as often as we please; but very few numbers are *powers* of other numbers.

410. *Signs.* The character $\sqrt{\ }$ is the sign of Evolution; thus, $\sqrt{64}$ = the square root of 64, $\sqrt[3]{125}$ = the cube root of 125, $\sqrt[5]{64}$ = the fifth root of 64.

A number is called a square number when it has an integer number for its root. Thus: 25 is a square number, because $\sqrt{25} = 5$, and $\sqrt{25}$ is a *rational* quantity.*

A number whose root cannot exactly be ascertained is not a square number, and its root is called an *irrational quantity* or *surd*. Thus: 12 is not a square number, because no number can be found equal to $\sqrt{12}$, and the quantity $\sqrt{12}$ is a surd or irrational quantity.

Observation.—It is evident that a surd is a number not expressible in figures, and hence that no surd multiplied or divided by any figures, whether integral or fractional, can give a rational number as the product or quotient.

SECTION I.—INVOLUTION TO THE SECOND POWER, OR FORMATION OF THE SQUARES OF NUMBERS.

411. All the operations of Involution rest upon an axiom in Multiplication (72). *Multiplication is always effected between two factors, if each of the parts of the one is multiplied by each of the parts of the other, and the sum of these products be taken. Hence—*

412. *If a number be divided into any two parts, the square*

* *Rational* is derived from *ratio*, used in the sense explained on page 181. Numbers are *rational* when their *ratio* or relation to other numbers can be exactly represented. Hence every number which can be expressed in figures is rational. But $\sqrt{12}$ or $\sqrt{3}$ is irrational, because no figures whatever could represent the ratio here intended. Irrational or incommensurable quantities were called by the Greeks ἄλογα, because their λόγος, or ratio, could not be stated.

of the number is equal to the product of the whole and one part, plus the product of the whole and the other part.

Demonstrative Example.—Because $18 = 10 + 8$;

$$\therefore\ 18 \times 18 = (18 \times 10) + (18 \times 8).$$

General Formula.—If $a = b + c$, $aa = ab + ac$.

413. *If a number be divided into two parts, the product of the whole and one part is equal to the square of that part, plus the product of the two parts.*

Demonstrative Example.—Because $18 = 10 + 8$;

$$\therefore\ 18 \times 10 = (10 \times 10) + (10 \times 8);$$

$$\text{and } 18 \times 8 = (8 \times 8) + (10 \times 8).$$

General Formula.—If $a = b + c$, $ab = bb + bc$, and $ac = cc + bc$.

Exercise CXI.

Divide each of the following numbers into two parts, and show how they illustrate the last two propositions:—

11, 15, 8, 24, 50, 19, 100, 26, 327, 48, 59, 25, 46, 72.

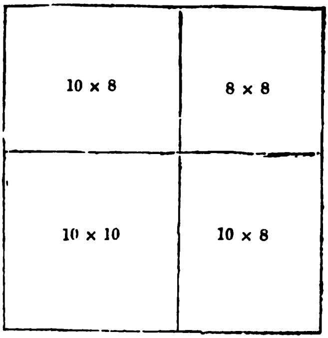
414. *If a number consist of two parts, the square of that number consists of the squares of those two parts, together with twice their product.**

Demonstrative Example.—For by (412) 18×18 was resolved into $(18 \times 10) + (18 \times 8)$.

And by (413) 18×10 was resolved into $(10 \times 10) + (8 \times 8)$, and 18×8 into $(8 \times 8) + (8 \times 10)$.

Therefore $18 \times 18 =$
$(10 \times 10) + (8 \times 8)$
$+ (10 \times 8) + (10 \times 8)$
$= 10^2 + 8^2 + 2\,(10 \times 8)$.

General Formula.—
Whenever $a = b + c$;
then $a^2 = b^2 + c^2 + 2bc$.



* The annexed diagram will illustrate this. For let the line be divided into the two parts, 10 and 8, it is evident that the whole area of the square described upon that line is made up of the parts 10^2, 8^2, and $2\,(10 \times 8)$.

EXERCISE CXII.

Divide each of the following numbers into two parts, and show how their squares are formed:—

15, 18, 24, 65, 23·5, 49, 10·8, 796, 235, 47, 32, 56, 71.

415. *Corollary.—The square of any number is equal to four times the square of half that number.*

Demonstrative Example.—The product of two equal numbers is the square of either of them, and because $18 = 9 + 9$, then by (414)—

$18 \times 18 = (9 \times 9) + (9 \times 9) + 2\,(9 \times 9)$, or $18^2 = 4\,(9^2)$.

General Formula.—If $a = 2b$, then $a^2 = 4b^2$.

EXERCISE CXIII.

Verify each of the last four propositions by 12 different numerical examples of your own selection.

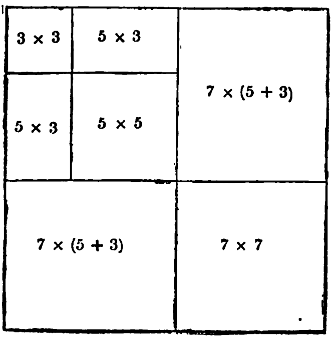
416. *If a number be divided into three parts, the square of the whole is made up of the square of the first + twice the product of the first into the sum of the second and third; the square of the second + twice the product of the second and third; and the square of the third.*

Demonstrative Example.—Divide 15 into 7, 5, and 3.*

Consider the 5 and 3 as one part and indicate it thus, $\overline{5 + 3}$.

Then (414) 15^2 or $(7 + 5 + 3)^2 = 7^2 + 2\,(7 \times \overline{5 + 3}) + (5 + 3)^2$.

But by (414) $(5 + 3)^2 = 5^2 + 2\,(5 \times 3) + 3^2$.



Hence

15^2 or $(7 + 5 + 3)^2 =$

$7^2 + (2 \times 7 \times \overline{5 + 3})$

$+ 5^2 + (2 \times 5 \times 3) + 3^2$.

$= 49 + 112 + 25 + 30$

$+ 9 = 225$.

* The annexed figure will illustrate this. For let the line be divided into three parts, the area of the square described upon it is evidently made up of seven portions, whose dimensions are those given in the text.

General Formula.—Let $a = b + c + d$;

then a^2 shall equal $b^2 + 2b\,(c + d) + c^2 + 2cd + d^2$.

For consider $\overline{c + d}$ as one part;

then by (414) $a^2 = b^2 + (2b \times \overline{c + d}) + (c + d)^2$.

But the last term, or $(c + d)^2 = c^2 + 2cd + d^2$.

Hence a^2 or $(b + c + d)^2 = b^2 + (2b\,\overline{c + d}) + c^2 + 2cd + d^2$.

Exercise CXIV.

Divide each of the following numbers into three parts, and show how they illustrate this proposition:—

6, 11, 18, 24, 39, 58, 111, 416, 72, 344, 718, 654, 419.

417. *If a number be divided into any number of parts, the square of the whole is made up of the squares of each of the parts, together with twice the product of each part into the sum of all the succeeding parts.*

Demonstrative Example.—Because $12 = 3 + 9$;

$\therefore$ by (414) $12^2 = 3^2 + (2 \times 3 \times 9) + 9^2$.

But $9 = 2 + 7$, and $9^2 = 2^2 + (2 \times 2 \times 7) + 7^2$.

Wherefore $12^2 = 3^2 + (2 \times 3 \times 9) + 2^2 + (2 \times 2 \times 7) + 7^2$.

But $7 = 4 + 3$ $\therefore 7^2 = 4^2 + (2 \times 4 \times 3) + 3^2$.

Wherefore $12^2 = 3^2 + (2 \times 3 \times 9) + 2^2 + (2 \times 2 \times 7) + 4^2 + (2 \times 4 \times 3) + 3^2$.

Or $(3 + 2 + 4 + 3)^2 = 3^2 + [2 \times 3 \times \overline{2 + 4 + 3}] + 2^2 + [2 \times 2 \times \overline{4 + 3}] + 4^2 + [2 \times 4 \times 3] + 3^2$.

For it is obvious that if we first divide a number into two parts and apply proposition (414), then again divide one of those parts into two and apply the same proposition, and so on; we may expand the square of a number in this way without limit, and the result will always take the same form.

General Formula.—If $a = b + c + d + e + f$;

Then $a^2 = b^2 + 2b\,(c + d + e + f) + c^2 + 2c\,(d + e + f) + d^2 + 2d\,(e + f) + e^2 + 2ef + f^2$.

EXERCISE CXV.

Verify each of the following assertions :—

Because $20 = 9 + 6 + 3 + 2$, $\therefore 20^2 = 9^2 + (2 \times 9 \times 11) + 6^2 + (2 \times 6 \times 5) + 3^2 + (2 \times 3 \times 2) + 2^2$.

$2738^2 = 2000^2 + (2 \times 2000 \times 738) + 700^2 + (2 \times 700 \times 38) + 30^2 + (2 \times 30 \times 8) + 8^2$.

EXERCISE CXVI.

Divide each of the following numbers into not less than four parts, and show how its square is made up :—

4, 13, 25, 41, 38, 65, 312, 417, 982, 645, 123, 444, 7098.

418. It follows from the laws of our system of notation* that—

The square of any number cannot contain more than twice as many figures as are contained in the number itself, nor less than twice as many, minus one.

Demonstrative Example.—The highest number which can be written with 3 digits only is 999. The square of this number must be less than the square of 1000. But $1000^2 = 1000000$, and this is the smallest number which extends to the seventh place. Wherefore the square of 999 cannot contain more than 6 digits. Again: the lowest number which requires 3 figures to express it is 100, and $100^2 = 10000$. But this number extends to the fifth place. Wherefore the square of any number consisting of 3 digits cannot contain more than 6 nor less than 5 digits.

In the same manner it might be proved that—

The square of a number of 2 places cannot have more than 4 or less than 3 digits.

The square of a number of 4 places cannot have more than 8 or less than 7 digits.

The square of a number of 5 places cannot have more than 10 or less than 9 digits.

And, generally, if a number contain n figures its square contains not more than $2n$ and not less than $2n - 1$.

* The student should notice the phraseology here employed. The fact here stated is not a fundamental principle of the Science of Arithmetic, but one of the consequences of our having adopted a peculiar system of notation.

** SECTION II.—THEORY OF NUMERICAL SQUARES AND PRODUCTS (*continued*).

419. The theorems in this section are intended still further to illustrate the nature of a square number, and the manner in which it may be formed. But they are not necessary for the comprehension of the ordinary rule for extracting the square root, and may be passed over by the student who is reading this book for the first time.

420. *The square of the product of two or more numbers is the same as the product of their squares.*

Demonstrative Example.—$(5 \times 8)^2 = 5^2 \times 8^2 = 1600$.

For by (65) factors may be multiplied in any order, therefore $(5 \times 8)(5 \times 8)$, or $5 \times 8 \times 5 \times 8 = 5 \times 5 \times 8 \times 8$, or $5^2 \times 8^2$.

General Formula.—$(ab)^2 = a^2b^2$, because $abab = aabb$.

421. *Corollary.—If one number measures another its square measures the square of that other.*

Demonstrative Example.—Because 3 is a measure of 12, 3^2 is a measure of 12^2, for $3 \times 4 = 12$; therefore (420) $3^2 \times 4^2 = 12^2$, and 3^2 is a measure of 12^2.

General Formula.—If a be a measure of b, a^2 is a measure of b^2.

For let a be contained in b m times, then $am = b$, and $a^2m^2 = b^2$. Wherefore a^2 is a measure of b^2.

EXERCISE CXVII.

Resolve each of the following numbers into two or more factors, and prove that the square of the products equals the product of the squares:—

6, 15, 27, 48, 50, 240, 168, 144, 36, 25, 72.

422. *The product of the sum and difference of any two numbers equals the difference of their squares.*

Example.—Let the numbers be 12 and 8; their sum is 20 $(12 + 8)$ and their difference 4 $(12 - 8)$.

$(12 + 8) \times (12 - 8) = 12^2 - 8^2$; *i.e.*, $20 \times 4 = 144 - 64$.

General Formula.—$(a + b) \times (a - b) = a^2 - b^2$.

423. *Corollary I.—If there be any two numbers whose difference is one, the difference of their squares equals their sum.*

Example.—Take the series of numbers—

1, 2, 3, 4, 5, 6, 7, 8, 9, &c.

The difference between the square of 1 and the square of 2 = 1 + 2; the difference between the square of 2 and the square of 3 = 2 + 3; between 4^2 and $5^2 = 4 + 5$, and so on.

General Formula.—By (414), $(a + b)^2 = a^2 + 2ab + b^2$.

$$\therefore (a + 1)^2 = a^2 + 2a + 1^2.$$

$$\therefore (a + 1)^2 \text{ exceeds } a^2 \text{ by } 2a + 1.$$

Hence the difference between the squares of any two consecutive numbers = twice the less + 1, and this is necessarily the sum of the two numbers. Thus 378^2 exceeds 377^2 by 2 (377) + 1, or 755. But this is 378 + 377.

424. *Corollary II.—If there be two fractions which added together make one, the difference of their squares is the difference of the fractions themselves.*

For in this case the product of the sum and difference = the difference, the sum being unity.

Example.—Because $\frac{3}{4} + \frac{2}{8} = 1$, $(\frac{3}{4})^2 - (\frac{2}{8})^2 = \frac{3}{4} - \frac{2}{8}$.

Or $\frac{9}{16} - \frac{4}{64} = \frac{36}{64} - \frac{4}{64} = \frac{32}{64} = \frac{1}{2}$, which is the difference between $\frac{3}{4}$ and $\frac{2}{8}$.

Under no other circumstances can the difference between the squares of two fractions be the same as between the fractions themselves.

Exercise CXVIII.

Take the sum and difference of the following pairs of numbers, and show that the product equals the difference of their squares:—

1. 5 and 9; 28 and 10; 11 and 12; 14 and 17; 15 and 9.
2. 6 and 7; 8 and 15; 23 and 24; 100 and 150; 65 and 85.
3. $\frac{1}{4}$ and $\frac{1}{3}$; $\frac{2}{7}$ and $\frac{4}{9}$; $\frac{5}{12}$ and $\frac{14}{24}$; ·47 and ·01; 1·25 and ·16.

425. *The square of the difference between any two numbers equals the sum of their squares, minus twice their product.*

Example.—$(12 - 7)^2 = (12 - 7)(12 - 7)$.

Now $(12 - 7)(12 - 7) = 12^2 - 2(12 \times 7) + 7^2$.

$$i.e.,\ 5 \times 5 = 144 + 49 - 168 = 25.$$

General Formula.—$(a - b)(a - b) = a^2 - 2ab + b^2$.

Exercise CXIX.

Take the difference between the two numbers of each of the following pairs, and show how this truth may be illustrated:—

1. 12 and 4; 17 and 6; 2 and 3; 14 and 17.
2. 20 and 14; 53 and 55; 100 and 9; 27 and 13.
3. $\frac{1}{3}$ and $\frac{1}{4}$; $\frac{2}{9}$ and $\frac{3}{7}$; $\frac{4}{11}$ and $\frac{5}{17}$; $\frac{16}{18}$ and $\frac{3}{8}$.
4. ·25 and 1·2; 408 and 5·6; 11·1 and 111; 18·3 and ·002.

426. From propositions (414) and (425) we find that—

The square of the sum of two numbers exceeds the sum of their squares by twice their product.

The square of the difference of two numbers is less than the sum of their squares by twice their product.

We have, therefore, in all cases in which two numbers are taken, three magnitudes—

$$(7 + 5)^2 \qquad 7^2 + 5^2 \qquad (7 - 5)^2$$
$$(a + b)^2 \qquad a^2 + b^2 \qquad (a - b)^2,$$

of which the first exceeds the second as much as the second exceeds the third. Wherefore by (54) the sum of the first and third equals twice the second, or $(a + b)^2 + (a - b)^2 = 2\,(a^2 + b^2)$, or *the square of the sum of two numbers, together with the square of their difference, make up twice the sum of their squares.* Hence also the following corollary may be deduced.

427. *Corollary II.—The square of the sum of two numbers exceeds the square of their difference by four times their product.**

Demonstrative Example.—For $(10 + 3)^2$ exceeding $10^2 + 3^2$ by twice the product of 10 and 3, and $10^2 + 3^2$ exceeding $(10 - 3)^2$ by twice their product also, therefore $(10 + 3)^2$ must exceed $(10 - 3)^2$ by 4 times their product.

General Formula.—$(a + b)^2 - (a - b)^2 = 4ab.$

Exercise CXX.

Verify the last two Corollaries by taking the squares of the sum and difference of the following pairs of numbers:—

1. 7 and 12; 4 and 9; 18 and 7; 17 and 13.
2. 16 and 20; 14 and 16; 24 and 20; 15 and 25.
3. 10·5 and 8·5; 12·1 and 50·9; 1·6 and 1·8; 40 and 45.

* *This proposition* is analogous to Euclid's 8th proposition, Book ii.

428. *Corollary III.—Four times the square of half the sum of any two numbers exceeds four times the square of half their difference by four times their product.*

Demonstrative Example.—Because (427) $(10 + 6)^2 - (10 - 6)^2 = 4 \times 10 \times 6$;

And by (415) $(10+6)^2 = 4\left(\frac{10+6}{2}\right)^2$ and $(10-6)^2 = 4\left(\frac{10-6}{2}\right)^2$;

$$\therefore\ 4\left(\frac{10+6}{2}\right)^2 - 4\left(\frac{10-6}{2}\right)^2 = 4 \times 10 \times 6.$$

General Formula. $4\left(\frac{a+b}{2}\right)^2 - 4\left(\frac{a-b}{2}\right)^2 = 4ab.$

If now we take a fourth part of every one of these quantities we have the following result:—

429. *Corollary IV.—The square of half the sum of two numbers exceeds the product of those numbers by the square of half their difference.**

Example. $\left(\frac{10+6}{2}\right)^2 - \left(\frac{10-6}{2}\right)^2 = 10 \times 6.$

$$\begin{aligned}
10^2 &= 11 \times 9 + 1^2 \\
10^2 &= 12 \times 8 + 2^2 \\
10^2 &= 13 \times 7 + 3^2 \\
10^2 &= 14 \times 6 + 4^2 \\
10^2 &= 15 \times 5 + 5^2 \\
10^2 &= 16 \times 4 + 6^2 \\
10^2 &= 17 \times 3 + 7^2 \\
10^2 &= 18 \times 2 + 8^2 \\
10^2 &= 19 \times 1 + 9^2
\end{aligned}$$

Because $10 + 10 = 20$, the product of any two numbers whose sum is 20, added to the square of half their difference, will equal 10^2.

General Formula.

$$\left(\frac{a+b}{2}\right)^2 - ab = \left(\frac{a-b}{2}\right)^2.$$

* This is the principle on which the ordinary mental rule for squaring a number is founded. The rule is, "Add the lower unit to the upper, multiply by the tens, and add the square of the unit." Thus 34×34, by adding the lower unit to the upper, becomes 38×30, or 1140. But this product of two numbers is less than the square of half their sum (34), by the square of half their difference (4). Hence $38 \times 30 + 4^2$, or $1140 + 16$, or $1156 = 34^2$.

When the unit is greater than 5 it will be more convenient to put the rule into another form: "Take from the one unit a number which will make the other an even *ten*; multiply these two numbers together, and add the square of the number which was subtracted," *e.g.*, 67^2. Add 3 to one 67 and take it from the other, then 70×64, or 4480, is the product to be obtained. But this product of 70×64 is less than the square of 67, or half their sum, by the square of 3, or half their difference, and $70 \times 64 + 3^2$, or $4480 + 9$, or $4489 = 67^2$.

By the other method $74 \times 60 + 7^2$, or $4440 + 49 = 4489 = 67^2$.

Corollary IV. is analogous to the 5th proposition of Euclid, Book ii.

EXERCISE CXXI.

Make up the square of each of the following numbers in four different ways, by the help of the last corollary :—

1.	7 ;	12 ;	15 ;	20.
2.	5 ;	16 ;	8 ;	24.
3.	14·4 ;	26 ;	30·5 ;	84·7.
4.	$\frac{3}{4}$;	$7\frac{1}{3}$;	$16\frac{4}{5}$;	$18\frac{1}{8}$.

430. The truth of the following propositions may easily be inferred from the foregoing. They should be verified in each case by the use of other numbers.

1. $\left(\frac{A + B}{2}\right)^2 + \left(\frac{A - B}{2}\right)^2 = \frac{A^2 + B^2}{2}$

   or $\left(\frac{7 + 5}{2}\right)^2 + \left(\frac{7 - 5}{2}\right)^2 = \frac{7^2 + 5^2}{2}$.

2. $(A + B)^2 + (A + B)(A - B) = 2A(A + B)$

   or $(7 + 5)^2 + (7 + 5) \times (7 - 5) = 2 \times 7(7 + 5)$.

3. $(A - B)^2 + (A + B)(A - B) = 2A(A - B)$

   or $(7 - 5)^2 + (7 + 5) \times (7 - 5) = 2 \times 7(7 - 5)$.

4. $(A + B)^2 - (A + B)(A - B) = 2B(A + B)$

   or $(7 + 5)^2 - (7 + 5) \times (7 - 5) = 2 \times 5(7 + 5)$.

5. $(A + B)(A - B) - (A - B)^2 = 2B(A - B)$

   or $(7 + 5) \times (7 - 5) - (7 - 5)^2 = 2 \times 5(7 - 5)$.

6. $(A + B)B + \left(\frac{A}{2}\right)^2 = \left(\frac{A}{2} + B\right)^2$

   or $(6 + 4)4 + \left(\frac{6}{2}\right)^2 = \left(\frac{6}{2} + 4\right)^2$.

7. $(A + B)^2 + B^2 = 2(A + B)B + A^2$

   or $(7 + 5)^2 + 5^2 = 2(7 + 5)5 + 7^2$.

8. $(A + B)^2 + A^2 = 2(A + B)A + B^2$

   or $(7 + 5)^2 + 7^2 = 2(7 + 5)7 + 5^2$.

EXERCISE CXXII.

(a). Express each of the foregoing propositions in words, and give six numerical illustrations of each.

(b). Make four numerical illustrations of each of the propositions in this section.

SECTION III.—EVOLUTION FROM THE SECOND POWER, OR EXTRACTION OF THE SQUARE ROOT OF NUMBERS.

431. As in Division and other parts of Arithmetic, so here, it is necessary to break up every sum into such smaller sums as shall come within the range of our tables; and in fact to do to a number part by part what we wish to do to the whole. In all former rules, however, we have found that whatever was done to the parts successively was done to the whole; but in extracting the roots of numbers this is not the case; for, *by finding the roots of the several parts which compose a number, we do not find the root of the whole number;* if it were so, the squares of these several parts added together would equal the square of the whole number, which by (414) is impossible.

432. *If we have a number whose root when extracted will consist of two parts, the original number must contain not only the squares of those two parts, but twice their product also.*

This is only another form of the truth stated in (414), it is here put in the form adapted to the inverse process of extracting the root of a given number, and needs no demonstration.

443. *Example I.*—Find the square root of 25.

$$
\begin{array}{rl}
25 & \\
\underline{9} & = 3^2 \\
16 & \\
\underline{12} & = 2\ (2 \times 3) \\
4 & \\
\underline{4} & = 2^2 \\
\cdot\ \cdot &
\end{array}
$$

We first choose a number, 3, whose square is certainly contained in 25; on taking this away we observe 16 remain. Now if we were to take the square root of this number, which is 4, and add it to the 3, we should be clearly wrong, for the square root of $(9 + 16)$ is not $(3 + 4)$ any more than the square root of $\overline{a^2 + b^2}$ is $\overline{a + b}$; but since a number, if it contains the square of the sum of two others, must contain the sum of their squares *and twice their product* (414), we must be able to take from the 16, not only the square of the new part, but also twice the product of it, and the other part. Now if we choose the number 2, we observe that its square (4), and twice the product of itself, and the first found number, 3, will make up 16, $2^2 + 2\ (2 \times 3) = 16$. Hence the number 25 has had taken from it in succession, the square of 3, twice the product of 3 and 2, and the square of 2. ∴ 25 contains the square of $(3 + 2)$, for $(3 + 2)^2 = 3^2 + (2 \times 3 \times 2) + 2^2$.

434. *Example II.*—Extract the second root of 1225.

$$\begin{array}{l} 1225 \\ \underline{900} = 30^2 \\ 325 \\ \underline{325} = 2 \times 30 \times 5 + 5^2 \\ \ldots \end{array}$$

We observe that there are 12 hundreds here. Now the nearest number of hundreds whose root can be easily ascertained, is 900, which is the square of 30. But the root of 1225 must be greater than 30, because 325 remain; let the other part of the root be a, then the root is $(30 + a)$; if so, then $1225 = 30^2 + (2 \times 30a) + a^2$. As 30^2 has already been taken away, the remaining 325 must contain twice the product of 30 into the new part, together with the square of the new part, $(2 \times 30), a + a^2$, or $(2 \times 30 + a)\ a$. If therefore we double the 30, choose a new part, 5, and add to it, and then multiply the 65 by the new part, this product, 325, will equal $(2 \times 30 \times 5) + 5^2$, and as this is the same as the remainder, the whole number $1225 = (\overline{30 + 5})^2$.

435. *If we divide any number into portions by pointing off every second figure, beginning with the unit, the number of the points thus made will show how many figures are contained in the root.*

Demonstrative Example.—The root of $\acute{5}|6\acute{2}|1\acute{8}$ will contain three figures; for by (421) the first portion, 50000, will have for its root a number of three places, the second portion, 6200, will have a number of the second place, and the third, 18, will have a unit for its root.

Hence, to find what will be the place of the first figure in the root it is usual to place a point over the first, the third, the fifth, and every alternate place in the number; and the number of these points always shows how many figures are in the root.

436. *If the square root of a number consist of several parts, the number itself contains the square of each part together with twice the product of that part into the sum of all the preceding parts.*

This is only the same proposition as (417) expressed in a form adapted for evolution, and needs no demonstration.

437. *Example III.*—Extract the square root of 613089. On pointing this number (435), $\dot{6}1\dot{3}0\dot{8}9$, we first observe that there will be three figures in the root, *i. e.*, that it will consist of a number of hundreds; 610000 is therefore the first part of the number selected. Now because $\sqrt{61}$ is more than $\sqrt{49}$ and less than $\sqrt{64}$, the root of 61 tens of thousands must be more than 700 and less than 800. 700 is therefore the first part of the root, and on subtracting its square we find that 123089 remains. We next have to find how many tens are in the root. Whatever this number is, its square together with twice the product of itself and 700, must be contained in the remainder, *i.e.*, $2 \times 700 \times$ the number of tens $+$ the square of the number of tens; or if

$$\begin{array}{rll} & 6'13'08'9 & - 700 + 80 + 3 = 783 \\ 700^2 = & \underline{490000} & \\ & 123089 & \\ (1400 + 80)\,80 = & \underline{118400} & = 2 \times 700 \times 80 + 80^2 \\ & 4689 & \\ (1560 + 3)\,3 = & \underline{4689} & = 2 \times 3 \times \overline{700 + 80} + 3^2 \\ & \cdots\cdot & \end{array}$$

x be the number of tens $(1400 + 10x)\,10x$. Now on applying 14000 to the remainder, we see that 8 is the nearest quotient, we therefore take 80 as the second part of the root. After subtracting the square of this part, together with twice the product of itself and the former part, we find 4689 remaining. We have now to find the third part of the root, and by (436) this must be such a number that its square + twice the product of itself and the sum of the preceding parts shall be contained in 4689, *i. e.*, 2 (780 $\times$ the new number) + the square of the new number. But $2 \times 780 = 1560$, and if we apply this to 4689 we have the quotient 3. Wherefore 2 (783) + 3^2, or 1563×3, or 4689, should be contained in the remainder.

438. It was stated in (416) that $(a + b + c)^2 = a^2 + 2\,a\,(b + c) + b^2 + 2\,bc + c^2$. Now the three parts of the number we have here found are 3, 80, and 700, and we have taken away—

$$3^2 + 2 \times 3 \times \overline{700 + 80} + 80^2 + 2 \times 700 \times 80 + 700^2.$$

$$\text{Hence} \quad \sqrt{613089} = \overline{700 + 80 + 3} = 783.$$

In this case the number proposed is an exact square, but in most cases a remainder is left after the operation, and we are only able to find the *nearest* square root.

439. The square root of a fraction is always to be found by taking the square root of the numerator for a new numerator, and the square root of the denominator for a new denominator.

For the square of a fraction is to be found by multiplying their numerators and denominators severally.

If $\frac{5}{7} \times \frac{5}{7}$, or $\left(\frac{5}{7}\right)^2 = \frac{25}{49}$, the square root of $\frac{25}{49}$ is $\frac{\sqrt{25}}{\sqrt{49}} = \frac{5}{7}$.

440. If the numerator and denominator are both multiplied by the denominator, the root of the fraction may always be so found as to give a rational denominator.

For let it be required to find the root of $\frac{15}{7}$ or $\frac{\sqrt{15}}{\sqrt{7}}$.

$$\frac{15}{7} = \frac{15 \times 7}{7 \times 7} = \frac{15 \times 7}{7^2} \therefore \sqrt{\frac{15}{7}} = \sqrt{\frac{15 \times 7}{7^2}} = \frac{\sqrt{15 \times 7}}{7}.$$

441. A fraction may have a rational square root, and yet appear to have none; *e.g.*, $\frac{75}{147}$: the roots of the numerator and denominator cannot be found; but if we divide both by 3 the fraction becomes $\frac{25}{49}$, the root of which is rational. Hence, before we attempt to extract the square root of a fraction, we must reduce it to its lowest name. The root of such a fraction must always be a fraction, and cannot be a whole number, for no whole number could have a fraction for its square.

442. In order to avoid a double operation, it is usual to reduce every vulgar fraction whose square is to be found, to a decimal form first, and then to apply the following principle:—

443. The roots of numbers which are expressed decimally can always be found readily when the expression extends to an even number of decimal places.

Demonstrative Example.—Because (250) the unexpressed denominator of 791·82 is 100, the root of 791·82 consists of $\sqrt{79182} \div \sqrt{100}$, *i.e.*, $\sqrt{791{\cdot}82} = \frac{\sqrt{79182}}{\sqrt{100}} = \frac{\sqrt{79182}}{10}$, therefore if we find the root of 79182, neglecting the decimal, and then mark off the unit *figure* of the answer by a decimal point (*i.e.*, divide it by 10) we

shall have the root of the fraction. But had the fraction been 7918·2, or $\frac{79182}{10}$, the sum could not easily be solved, for $\sqrt{10}$ is an irrational quantity. Similar reasoning gives us the following results :—

444. The square root of a number of 2 decimal places has 1 decimal place.

The square root of a number of 4 decimal places has 2 decimal places.

The square root of a number of 6 decimal places has 3 decimal places.

The square root of a number of 8 decimal places has 4 decimal places.

Therefore if in any Evolution sum we want accuracy to a certain number of decimal places, we must have twice as many decimal places in the sum.

Whenever a number has an odd number of decimal places we may add a cipher—and by (253) this will not alter its value—and then, remembering the above rule, may neglect the decimal points, and proceed as in whole numbers.

445. *Example.*—Find the square root of 2710·382.

If we consider this whole number, without the decimal point, as the numerator of a fraction whose denominator is 1000, we shall not be able to express the answer decimally, for $\sqrt{1000}$ is not a suitable number for the denominator of a decimal fraction. But because $\sqrt{10000} = 100$, the root of 2710·3820, or of $\frac{27103820}{10000}$, will have 100 for its denominator.

$$\begin{array}{rr l}
 & 2710{\cdot}3820 & 5000 + 200 + 6 = 5206 \\
5000^2 = & \underline{2500{\cdot}0000} & \\
 & 210{\cdot}3820 & \\
10200 \times 200 = & \underline{204{\cdot}0000} & = (2 \times 5000 \times 200) + 200^2 \\
 & 6{\cdot}3820 & \\
10406 \times 6 = & \underline{6{\cdot}2436} & = 2 \times (5000 + 200) \times 6 + 6^2 \\
 & 13{\cdot}84 &
\end{array}$$

5206 is the nearest root of 27103820, but as this latter number is the numerator of a fraction whose denominator is 10000, the denominator of its root is 100, and 52·06 is the nearest answer. It is evident that the root might be carried to a greater degree of accuracy if other pairs of ciphers were added to the number, for every two places added to the number would add one to the root.

RULE FOR EXTRACTING THE SQUARE ROOT.

446. Place a point over the unit and over every second figure from it in both directions. If there be any decimal places, add a cipher if necessary, so that the last figure shall be pointed.

Find the nearest root of the first period, and subtract it. Add the next period to the remainder. Double the first part of the root, and find how many times it is contained in this new dividend, omitting the last figure. Add the quotient thus found to the divisor, multiply the divisor thus formed by the second figure of the quotient, subtract this product as before, and add the next period to the remainder. Proceed in this way until the last period has been brought down.

447. *Observation.*—In the examples just given we have set down more figures than are necessary in the working. The following example will show the ordinary process, and also the extended process, of which it is an abridgment.

Example I.—Extract the root of 141376.

Complete or uncontracted process.

$$\begin{array}{rrl} & 1\acute{4}1\acute{3}7\acute{6} & 300 + 70 + 6 \\ 300^2 = & \underline{90000} & \\ & 51376 & \\ 670 \times 70 = & \underline{46900} & = (2 \times 300 \times 70) + 70^2 \\ & 4476 & \\ 746 \times 6 = & \underline{4476} & = 2 \times (300 + 70) \times 6 + 6^2. \\ & \cdots & \end{array}$$

Ordinary or uncontracted process.

$$\begin{array}{rl} & 1\acute{4}1\acute{3}7\acute{6}\,(376 \\ & \underline{9} \\ 67) & 513\,(7 \\ & \underline{469} \\ 746) & 4476\,(6 \\ & \underline{4476} \\ & \cdots \end{array}$$

$\sqrt{141376} = 376.$

It is evident that—

Because 141376 has been found to contain

$300^2 + (2 \times 300 \times 70) + 70^2 + (2 \times \overline{300 + 70} \times 6) + 6^2$,

$\therefore$ $141376 = (300 + 70 + 6)^2 = 376^2$.

448. *Example II.*—Find the square root of 7 to 3 places of decimals.

$$\begin{array}{l|l|l|l}
7 & 2{\cdot}645 & & \\
\underline{4} & 46 & 524 & 5285 \\
300 & 6 & 4 & 5 \\
\underline{276} & & & \\
\;\;2400 & & & \\
\;\;\underline{2096} & & & \\
\;\;\;\;30400 & & & \\
\;\;\;\;\underline{26425} & & & \\
\;\;\;\;\;\;3975 & & &
\end{array}$$

Observation.—It is evident that there is no limit to this process, for, by the addition of pairs of ciphers, the root of any number which is not a perfect square, can be ascertained to as many places of decimals as may be desired.

449. The method of extracting the root to a certain decimal place, is founded on a principle of somewhat wider application. If it be required to extract the root of a number a true to any fraction, say $\frac{1}{n}$, we may multiply the number by n^2, extract its root, and then divide the answer by n. For example, to find $\sqrt{59}$, so that the answer shall not differ from the truth by so much as $\frac{1}{12}$, we first multiply and divide 59 by 12^2. Then

$$\sqrt{59} = \sqrt{\frac{59 \times 12^2}{12^2}} = \sqrt{\frac{8496}{12^2}} = \frac{\sqrt{8496}}{12}.$$

But the nearest square root of 8496 is 92,

$\therefore$ $\sqrt{59} = \frac{92}{12} = 7\frac{2}{3}$ nearly.

Again, let it be required to find $\sqrt{31\frac{4}{7}}$ true to $\frac{1}{23}$.

$$31\tfrac{4}{7} = \tfrac{2}{7} = \frac{221 \times 23^2}{7 \times 23^2} = \tfrac{116909}{7} \div 23^2 = 16071\tfrac{2}{7} \div 23^2.$$

$$\text{Hence } \sqrt{31\tfrac{4}{7}} = \sqrt{16071} \div 23 = \tfrac{129}{23} = 5\tfrac{14}{23}.$$

So also $\sqrt{212}$ true to $\frac{1}{7} = \sqrt{\dfrac{212 \times 7^2}{7^2}} = \dfrac{\sqrt{10388}}{\sqrt{7^2}} = \frac{101}{7} = 14\frac{3}{7}$.

In the case of decimals, adding twice as many figures as are needed in the root, is in effect multiplying the numerator of the fraction by the square of that number which represents the degree of accuracy required. In the example just given, because $\sqrt{7}$ was found true to $\frac{1}{1000}$, the sum took this form—

$$\sqrt{7} = \sqrt{\frac{7 \times 1000^2}{1000^2}} = \frac{\sqrt{7000000}}{1000} = \frac{2645}{1000} = 2{\cdot}645.$$

The following rules will help to show when a number has and when it has not a rational square root:—

450. *No even number not divisible by 4 is a perfect square.*

For every even number may be represented by $2n$, its square therefore must always (420) be $4n^2$ and will always be divisible by 4.

451. *No odd number which, diminished by 1, is not divisible by 4, is a perfect square.*

Because every odd number may be expressed as $2n + 1$, the square of such a number must always be $4n^2 + 4n + 1$. This number diminished by unity is evidently divisible by 4.

452. *No number terminating in 2, 3, 7, or 8, is a perfect square.*

For if a square number does not terminate in a cipher, the unit figure in the square must have been obtained by squaring the unit figure of the root. And the square of each of the nine digits ends in 1, 4, 5, 6, or 9.

453. *A number ending in 5 cannot be a perfect square unless the number in the tens place be 2.*

For whenever 5 is the unit of a root, the square will always be 5^2 plus 2×5 multiplied into a certain number of tens; and 2×5, or 10, multiplied into tens will always give hundreds, and the square of such a number will consequently consist of 5^2 + a number of hundreds.

454. *No number terminated by an odd number of ciphers can be a perfect square.*

Because (444) a number ending with a cipher will always have twice as many ciphers at the end of its square.

455. *The square of every proper fraction must itself be a fraction, and cannot be a whole number.*

Because by (220) the product of two proper fractions is less than either of them.

456. *If the square of one number measures the square of another, the first number itself measures the second.*

For by (168) one number is measured by another when it contains all the prime factors of that other, and the square of a number contains no other prime factors than compose the number itself. Wherefore a^2 contains no other prime factors than are contained in a, and b^2 no others than are contained in b. Hence if a^2 measures b^2, b contains all the prime factors of a, and is therefore measured by it.

457. *If any fraction whatever be expressed in its lowest terms, its square cannot be a whole number.*

For if it could, then the square of $\frac{a}{b}$, or $\frac{a^2}{b^2}$, being a whole number, would have its numerator measured by its denominator; *i.e.*, b^2 would measure a^2 while b did not measure a, which would contradict the last proposition.

458. *If the square root of an integer number be not itself an integer, that root must be a surd or incommensurable quantity.*

For by (457) it cannot be a vulgar fraction, and therefore it cannot be a recurring decimal; because by (266) all recurring decimals can be exactly expressed as vulgar fractions.

459. *Observation.*—The rule for the extraction of the square root is not in any way dependent on the particular system of numeration with which we work. The same results would be found were the numbers expressed on any other scale of notation. Again, numbers which are perfect squares on the decimal system would be so also on all others, whereas the roots of 3, of 7, of 11, are surds, or incommensurable numbers, on all scales alike.

Exercise CXXIII.

Extract the square root of the following numbers:—

1. 17698849; 698485; 6084; 4096.
2. 841; 1287; 56821444; 714025.
3. 9585216; 45369; 10816; 745·29.
4. 3370·9636; 73008·04; ·000256; $\frac{2250}{3610}$.
5. 5·764801; 100996·84; ·047089.
6. 372344·04; 2768·412; 39705678.

7. 14590·2241 ; 9·8596 ; ·00591361.

8. 14·630625 ; 16·752649 ; ·006724.

9. $\frac{2}{3}$; $\frac{28}{29}$; $2\frac{1}{4}$; 15·8.

10. 6·4 ; ·64 ; ·064 ; ·0064 ; 640.

11. $12\frac{3}{5}$; $64\frac{1}{5}$; 18·27 ; 40·96.

12. $\frac{2·05}{2·25}$; $\frac{3·79}{·0016}$; $\frac{7·98}{52·4}$; $\frac{137}{219}$.

13. Find the square root of 11 true to $\frac{1}{130}$, and of 223 to $\frac{1}{40}$.

14. Find the square root of $79\frac{1}{8}$ true to $\frac{1}{20}$, and of $\frac{7}{13}$ to $\frac{1}{30}$.

15. Find the square root of 563 true to $\frac{1}{33}$, and of $413\frac{1}{9}$ to $\frac{1}{23}$.

16. Find $\sqrt{56}$ true to $\frac{1}{31}$, and $\sqrt{21\frac{3}{7}}$ true to $\frac{1}{11}$.

17. What is the difference between the square root of the sum of 1790 and 4451, and the sum of their square roots?

18. The product of two equal numbers is 509796. What are they?

SECTION IV.—INVOLUTION TO THE THIRD POWER, OR FORMATION OF THE CUBES OF NUMBERS.

460. The cube or third power of a number is found by multiplying it by itself twice, or by finding the product of three equal factors.

Thus $4 \times 4 \times 4 = 64 = 4^3 =$ the cube of 4.

$a \times a \times a = a^3 =$ the cube of a.

In the lines of figures—

1	2	3	4	5	6	7	8	9	10
1	8	27	64	125	216	343	512	729	1000

the lower numbers are the *Cubes* or third powers of those above them, and reciprocally the numbers in the upper line represent the *Cube Roots* of those below them.

461. These results should be committed to memory, as a knowledge of them is needed throughout the Rule for the Extraction of the Cube Root.

EXERCISE CXXIV.

Find by multiplication the cubes of the following numbers:—

1. 17 ; 19 ; 30 ; 42 ; 16.

2. 18 ; 12 ; 3·7 ; 6·8 ; 4·15.

3. $2\frac{1}{4}$; $\frac{5}{9}$; $\frac{7}{12}$; $\frac{3}{13}$; $\frac{6}{11}$.

462. It is necessary here to consider a number consisting of two parts, and to find how its cube is made up of the cubes of those parts.

463. * *If a number be divided into two parts, the cube of that number is equal to the cube of the first, together with three times the square of the first multiplied by the second, together with three times the square of the second multiplied by the first, together with the cube of the second.*

Demonstrative Example.—Let 10 be divided into two parts, 6 and 4.

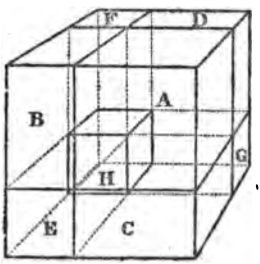
Then $(6 + 4)^3$ may be found thus:—

$$\begin{array}{ll}
6 + 4 & \\
6 + 4 & \\
\hline
6^2 + 6 \times 4 & = (6 + 4) \times 6 \\
\quad\;\; 6 \times 4 + 4 & = (6 + 4) \times 4 \\
\hline
6^2 + 2\,(6 \times 4) + 4^2 & = (6 + 4) \times (6 + 4) = 10^2 \\
6 + 4 & \\
\hline
6^3 + 2\,(6^2 \times 4) + 6 \times 4^2 & = (6 + 4)^2 \times 6 \\
\qquad\quad 6^2 \times 4 + 2\,(6 \times 4^2) + 4^3 & = (6 + 4)^2 \times 4 \\
\hline
\left.\begin{array}{c} 6^3 + 3\,(6^2 \times 4) + 3\,(6 \times 4^2) + 4^3 \\ \text{or} \\ 216 \;+\; 432 \;+\; 288 \;+\; 64 \end{array}\right\} & = (6 + 4)^3 = 10^3 = 1000
\end{array}$$

General Formula.—Let $a = b + c$, then

$$a^3 = b^3 + 3b^2c + 3bc^2 + c^3.$$

* The analogy between number and geometry is again worth noting here. For let each of the equal edges of a cube be similarly divided into two parts, and let lines be drawn from the points of section; then, if the cube be cut into portions at these lines, it will be found that the whole has been divided into eight parts, corresponding to the products mentioned in the text. If we call the whole line 12 and the longer and shorter portions of the equal lines be respectively 8 and 4, it may be seen that the eight portions are as follow:—



A, cube of 8, or 8^3.

B, C, and D, pieces 8 long, 8 wide, and 4 thick ($3 \times 8^2 \times 4$).

E, F, and G, pieces 8 long, 4 wide, and 4 thick ($3 \times 8 \times 4^2$).

H, cube of 4, or 4^3.

EXERCISE CXXV.

Divide the following numbers into 2 parts, and show how the cubes of the whole numbers are formed of the cubes of the parts:—

Example.—Because $5 = 3 + 2$,

$$\therefore 5^3 = 3^3 + (3 \times 3^2 \times 2) + (3 \times 3 \times 2^2) + 2^3.$$
$$125 = 27 + 54 + 36 + 8.$$

And because $26 = 20 + 6$,

$$\therefore 26^3 = 20^3 + (3 \times 20^2 \times 6) + (3 \times 20 \times 6^2) + 6^3.$$

1. 7; 9; 10; 12; 15.
2. 1·5; 63; 81; 24; 17.
3. 80; 21; 34; 13; 30.

464. *The difference between the cubes of any two numbers whose difference is* 1, *equals three times the square of the less, plus three times the less, plus one.*

For if in the former expression we substitute 1 for b,

$$\therefore (b + 1)^3 = b^3 + 3b^2 + 3b + 1.$$

Hence, for example, the difference between $(64)^3$ and $(65)^3 = 3 \times (64)^2 + (3 \times 64) + 1 = 12481$.

465. *Observation.*—From this it may be seen how great a distance there is between the cubes of any two consecutive numbers, and how few numbers are perfect cubes.

EXERCISE CXXVI.

(a). Verify the assertion in (464) in the case of the cubes of the first 10 numbers given in (460).

(b). Ascertain by this rule what is the difference between the cubes of the following pairs of numbers:—

1. 15 and 16; 21 and 22; 11 and 12; 17 and 18.
2. 30 and 31; 101 and 102; 24 and 25.
3. 125 and 126; 19 and 20; 46 and 47.

466. *The cube of a number cannot contain more than three times as many figures as the number itself, nor less than three times as many, minus two.*

Demonstrative Example.—It has been shown that the cube of 9, which is the highest number expressed by one digit, is 729, *i.e.*, it *extends* to three figures; while the cube of 10, the lowest number

of two digits, is 1000, a number of the fourth place. In the same manner it may be seen that every number between 10 and 100 will have its cube between 1000 and 1000000, and that—

A number of 1 digit has in its cube not more than 3 digits.

A number of two digits has in its cube not more than 6 digits, nor less than 4.

A number of 3 digits has in its cube not more than 9 digits, nor less than 7.

A number of 4 digits has in its cube not more than 12 digits, nor less than 10.

General Formula.—If a number contains n digits, its cube will contain not more than $3n$ and not less than $3n - 2$.

467. And because $\cdot 1 = \frac{1}{10}$ and $(\frac{1}{10})^3 = \frac{1}{1000} = \cdot 001$
$\cdot 01 = \frac{1}{100}$ and $(\frac{1}{100})^3 = \frac{1}{1000000} = \cdot 000001$
$\cdot 001 = \frac{1}{1000}$ and $(\frac{1}{1000})^3 = \frac{1}{1000000000} = \cdot 000000001$

Therefore the cube of a fraction in the first decimal place extends to the third decimal place.

The cube of a fraction in the second decimal place extends to the sixth decimal place.

And the cube of a fraction in the third decimal place extends to the ninth decimal place.

468. *The cube of the product of two or more numbers is the same as the product of their cubes.*

Demonstrative Example.—Because $12 = 3 \times 4$, therefore $12^3 = 3 \times 4 \times 3 \times 4 \times 3 \times 4$.

But (65) this product is the same with the factors in any order.

$$\therefore 12^3 = 3 \times 3 \times 3 \times 4 \times 4 \times 4 = 3^3 \times 4^3.$$

General Formula.—If $a = bcd$ then $a = b^3c^3d^3$.

Exercise CXXVII.

Resolve each of the following numbers into factors, and verify the assertion in (468) :—

1. 6; 10; 12; 15; 16.
2. 14; 21; 24; 30; 18.
3. 25; 32; 56; 100; 108.

SECTION V.—EVOLUTION FROM THE THIRD POWER, OR EXTRACTION OF THE CUBE ROOT OF NUMBERS.

469. *If a number has for its cube root a number consisting of two parts, it must contain the cube of the first part, plus three times the square of the first part multiplied by the second part, plus three times the product of the first part and the square of the second, plus the cube of the second.*

This is only the same truth that is given in (463) adapted to the inverse process of Evolution. It will suffice to explain the reason of the rule in all cases, even though the cube root contain three or more parts. For because $10^3 = 1000$, the cube of any number of tens will give the cube of that number multiplied by 1000 (468), *e. g.*, the cube of 70 or of $7 \times 10 = 7^3 \times 10^3 = 7^3 \times 1000$. If therefore we cut off the last three figures of any integer number, and find the cube root of the rest, the answer multiplied by 10 will give the cube root of that number of thousands. And because $100^3 = 1000000$, therefore the cube root of any number of millions equals the cube root of that number multiplied by 100. For by (468)

$$\sqrt[3]{27000000} = \sqrt[3]{27} \times \sqrt[3]{1000000} = \sqrt[3]{27} \times 100.$$

And if we cut off the last six figures of a number, and find the cube root of the rest, the answer will be a number which, multiplied by 100, will give the cube root of the millions.

Hence it follows that to find the cube root of any number which contains millions, we must first find the cube root of the millions and the thousands, and, having obtained it, consider this root, though consisting of two figures, as the first part, and then proceed to extract the root of the remaining part.

470. *If a point be made over the unit figure and over every third figure from it, the number of such points shows the number of digits in the cube root. e.g.,*

The cube root of $\dot{1}852\dot{8}91\dot{6}54\dot{2}$ will contain four digits.

The cube root of $7\dot{0}96\dot{2}$ will contain two digits.

The cube root of $18\dot{7}$ will contain one digit.

471. The following examples will show how this principle is employed in the extraction of the cube root:—

Example I.—

$$\begin{array}{r|l}
 & \dot{1}75\dot{6}16\ (50 + 6 \\
50^3 = & 125000 \\
\hline
3 \times 50^2 = 7500) & 50616 \\
\hline
7500 \times 6 = & 45000 \\
3 \times 50 \times 6^2 = & 5400 \\
6^3 = & 216 \\
\hline
 & 50616
\end{array}$$

On placing points it is observed that the answer will contain two figures, tens and units. We first find the nearest cube root of the first period, 175000. This gives 5, which is therefore 5 tens. On subtracting the cube of 50, 50616 is found to remain. But by (469) this number must contain three times the square of 50 multiplied by the new part, plus three times 50 multiplied by the square of the new part, plus the cube of the new part. In order to find, approximately, what this new part is, we divide 50616 by three times the square of 50. When the quotient, 6, has thus been obtained, it becomes necessary to subtract three several sums from the 50616, viz., $3 \times 50^2 \times 6$, which is 45000; then $3 \times 50 \times 6^2$, which is 5400; and lastly, 6^3, which is 216. But the sum of these numbers equals the former remainder, wherefore

$$56 = \sqrt{175616}.$$

For by (464) it has been shown that if $a = (b + c)$, then $a^3 = b^3 + 3b^2c + 3bc^2 + c^3$.

And the number 175616 has been shown to contain—

$$50^3 + (3 \times 50^2 \times 6) + (3 \times 50 \times 6^2) + 6^3.$$

Therefore $175616 = (50 + 6)^3 =$ and $56 = \sqrt[3]{175616}$.

472. *Example II.*—Find the cube root of 33698267.

$$\begin{array}{r|l}
 & \dot{3}369\dot{8}26\dot{7}\ (323 \\
3^3 = & 27 \\
\hline
3 \times 30^2 = 2700 & 6698 \\
3 \times 30 \times 2 = 180 & \\
2^2 = 4 & \\
\overline{2884} \times 2 = & 5768 \\
\hline
 & 930\ 267 \\
3 \times 320^2 = 307200 & \\
3 \times 320 \times 3 = 2880 & \\
3^2 = 9 & \\
\overline{310089} \times 3 = & 930\ 267 \\
\hline
 & \dots\ \dots
\end{array}$$

The answer here will contain three digits. The nearest cube root of the millions is found to be 3 (or 300). On subtracting 27 and bringing down the remaining thousands we have 6698. These are thousands, and the cube root of thousands being

tens we have next to find the number of tens in the root; let this number be x. Then (469) 6698 ought to contain $30^2 + 3x \times 30 \times 3x^2 + x^3$. But x being a common factor of these quantities it will be convenient to consider it as $\overline{(30^2 \times 3 + 30 \times 3x + x^2)}\, x$, as by this means we avoid three several multiplications by the same number. To find the required number we try $30^2 \times 3$, or 2700 into 6698; the nearest quotient is 2. Place the 2 as the answer in the tens place, as x. Then multiply it by the sum of 3×30^2, $3 \times 30 \times 2$, and 2^2. Subtract this number, and bring down the next period.

It has now been found that 33698267 contains the cube of 32 tens, or 320, and also 930267 besides. 320 may now be considered as the first part of the root, and we have to find the other part. This will be a unit figure, for it is evident that the number 3 would have been too great in the tens place, and therefore that the answer is less than 330 and more than 320. Let the unknown unit be y; then by (470) 930267 ought to contain $3 \times 320^2y + (3 \times 320y^2) + y^3$, or as y is a common factor $\overline{(3 \times 320^2 + 3 \times 320y + y^2)}\, y$. In order to find y we take as a trial divisor 3×320^2, which is 307200, and find how many times it is contained in 930267; the quotient seems to be 3. On taking from the last remainder these three several portions we find no remainder; 323 is therefore the exact root required.

For because generally (464) the cube of $\overline{w + x + y} =$
$(w + x)^3 + 3\,(w + x)^2\, y + 3\,(w + x)\, y^2 + y^3$.

And $(w + x)^3 = w^3 + 3w^2x + 3wx^2 + x^3$.

Therefore $(w + x + y)^3 =$
$w^3 + 3w^2x + 3wx^2 + x^3 + 3\,(w + x)^2\, y + 3\,(w + x)\, y^2 + y^3$.

And $(300 + 20 + 3)^3 = 300^3 + (3 \times 300^2 \times 20) + (3 \times 300 \times 20^2) + 20^3 + (3 \times 320^2 \times 3) + (3 \times 320 \times 3^2) + 3^3$.

Hence as 33698267 has been found to contain all these several parts— $\sqrt[3]{33698267} = 300 + 20 + 3 = 323$.

473. *The cube root of a fraction may be found by taking the cube roots of its numerator and denominator respectively.*

Demonstrative Example.—Because $\left(\frac{3}{5}\right)^3 = \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}$

$$\therefore \sqrt[3]{\frac{27}{125}} = \frac{\sqrt[3]{27}}{\sqrt[3]{125}} = \frac{3}{5}.$$

General Formula.—$\sqrt[3]{\dfrac{a}{b}} = \dfrac{\sqrt[3]{a}}{\sqrt[3]{b}}$.

474. As in the Extraction of the Square Root, it is usual to reduce all vulgar fractions whose cube root is required, into a decimal form, and to let them be so expressed that their denominators shall have a rational cube root.

From (466) it may be seen that the only powers of 10 which have a rational cube root are the third, the sixth, the ninth, the twelfth, &c., and that the cube root of a number extending to the third decimal place has itself one decimal place; that of a number extending to the sixth has two places, &c. Hence, before extracting the root of a decimal fraction, it is necessary to place three times as many ciphers as there are figures required in the root, and that the number of decimal places should always be a multiple of 3.

475. The method of evolving a root by approximation to any fraction required (449) is as applicable here as for the square root. For to find the cube root of any number true as far as any given fraction, say $\dfrac{1}{n}$, we may multiply the number by the cube of n, extract its root, and then divide this answer by n. If, for example, it is required to find the cube root of 15 true to $\frac{1}{12}$, the number 15 may take the form $\dfrac{15 \times 12^3}{12^3}$ or $\dfrac{25920}{12^3}$. Now as the nearest cube root of 25920 is 29, the root required is $\frac{29}{12}$, or $2\frac{5}{12}$.

To find the cube root of $37\frac{8}{13}$ true to $\frac{1}{20}$. Multiply $37\frac{8}{13}$ by 20^3.

$$\tfrac{489}{13} \times 20^3 = 489 \times \tfrac{8000}{13} = \tfrac{3912000}{13} = 300923\tfrac{1}{13}.$$

But $\sqrt[3]{300923} = 67$. Hence $\sqrt[3]{\frac{489}{13}} = \frac{67}{20}$, or $3\frac{7}{20}$ nearly.

In like manner the cube root of 47 true to $\frac{1}{20}$ =

$$\sqrt[3]{\frac{47 \times 20^3}{20^3}} = \tfrac{72}{20} = 3\tfrac{3}{5}.$$

And the cube root of $23\frac{7}{8}$ true to $\frac{1}{13}$ =

$$\sqrt[3]{\frac{191 \times 13^3}{8 \times 13^3}} = \tfrac{37}{13} = 2\tfrac{11}{13}.$$

And the cube root of $\frac{5}{7}$ true to $\frac{1}{30}$ =

$$\sqrt[3]{\frac{5 \times 30^3}{7}} \div 30 = \tfrac{13}{15}.$$

Questions are seldom given in this form, but the common method of treating all fractions as decimals is evidently founded on the same principle. Thus, if we are required to find $\sqrt[3]{25}$ true to two places of decimals, this means that the answer is not to differ from the truth by so much as $\frac{1}{100}$, we therefore multiply 25 by the cube of 100, when it becomes 25000000; the nearest root of this is 292; on dividing this by 100 we have 2·92 as the required answer. The process in principle is identical with the preceding.

For $\sqrt[3]{25}$ true to $\frac{1}{100} = \sqrt[3]{\frac{25 \times 100^3}{100^3}} = \sqrt[3]{25 \times 100^3} \div 100$ $= 292 \div 100 = 2{\cdot}92.$

In like manner $\sqrt[3]{3{\cdot}1415}$ true to $\frac{1}{100} = \sqrt[3]{\frac{3{\cdot}1415 \times 100^3}{100^3}} =$

$$\sqrt[3]{3141500} \div 100 = 1{\cdot}46.$$

To extract the cube root of a number—

RULE I.

476. Place a point over the unit and over every third figure from it both ways. The number of points shows the number of figures in the root.

Find the nearest cube root of the first period, set it down as the first figure of the root, subtract its cube from the first period and bring down the next period to the right of the remainder.

Take three times the square of the first found figure, considered as a number of tens, and ascertain how many times it is contained in this new dividend. The quotient will form the second figure of the root.

Take three times the square of the first part + three times the product of the first and the second + the square of the second. Multiply this sum by the second figure of the root, and subtract the result.

If more periods remain to be brought down, repeat the same process, considering that part of the root already found as a number of tens.

Example.—Extract the cube root of 7 to three decimal places.

		7 (1·912
		1
		6000
$3 \times 10^2 = 300$		
$3 \times 10 \times 9 = 270$		
$9^2 = 81$		
651×9	$=$	5859
		141000
$3 \times 190^2 = 108300$		
$3 \times 190 = 570$		
$1^3 = 1$		
108871×1	$=$	108871
		32129000
$3 \times 1910^2 = 10944300$		
$3 \times 1910 \times 2 = 11460$		
$2^2 = 4$		
10955764×2	$=$	21911528
		10217472

RULE II.

477. Proceed, as in the former rule, to find the first part of the root, and to find the second part by trial.

To three times the first found figure annex the second; multiply by the second, add this product to the trial divisor, and multiply the whole sum by the second figure of the root.

To find the next trial divisor, add together the former complete divisor, the sum that completed it, and the square of the second part.

Having thus found the third part of the root, add it to the right hand of three times the former parts; multiply this sum by the third part, add the result to the last trial divisor, and multiply by the third part. Proceed in the same manner with other parts.

Observation.—This is an abridged form of the same process as the last. The reason will be easily seen in the following example:—

Example.—Find the cube root of 4822855.

$$
\begin{array}{lrl}
 & & 4\dot{8}22\dot{8}55\dot{0}(364 \\
 & & 27 \\
\text{1st Trial divisor} \quad 3 \times 30^2 = & 2700 & 21228 \\
(a).\ (3 \times 30 \times 6) + 6^2 = & \underline{576} & \\
 & 3276 \times 6 = & \underline{19656} \\
 & \underline{36} & 1572550 \\
\text{2nd Trial divisor } (b).\ 3 \times 360^2 = & 388800 & \\
(c).\ (3 \times 360 \times 4) + 4^2 = & \underline{4336} & \\
 & 393136 \times 4 = & \underline{1572544} \\
 & & 6
\end{array}
$$

Exercise CXXVIII.

Find the cube roots of the following numbers:—

1. 32768; 1157·625; 247673152.
2. 32461759; 78610563187; 5·088448.
3. 1·29503; 4173·281; 135796·744.
4. 8108486729; 165795999168.
5. 1·25; 3; 2197583; $\cdot\dot{4}$.
6. 2599609375; 247791486041.
7. 356·702522688; 219038133·952.
8. 138·348848448; 51645·087424.
9. Find the cube root of $\frac{5}{7}$ true to $\frac{1}{16}$, and that of $31\frac{2}{3}$ true to $\frac{1}{23}$
10. Find $\sqrt{\frac{4}{9}}$ true to $\frac{1}{30}$, and $\sqrt{\frac{11}{13}}$ true to $\frac{1}{43}$.
11. Find $\sqrt{24\frac{1}{7}}$ true to $\frac{1}{15}$, and $\sqrt{273\frac{1}{4}}$ to $\frac{1}{32}$.
12. Find the approximate cube root of the following numbers in a decimal form:—
    (a). 729863; 527410; 6283·0542.
    (b). $\frac{3}{8}$; ·68; ·0654; 27·0298.
    (c). $17\cdot\dot{4}$; $18\cdot0\dot{3}\dot{6}$; $509\cdot\dot{7}$; $62\cdot8\dot{7}\dot{5}$.
13. The product of 3 equal numbers is 3189506048, what are they

---

$$
\begin{array}{rr}
(a) & 30 \\
 & \underline{3} \\
(3 \times 30) + 6 = & 96 \\
 & \underline{6} \\
(3 \times 30 \times 6) + 6^2 = & 576
\end{array}
$$

(*b*). This number, 388800, is clearly 3 times the square of 360, for it contains 3 times the square of 300 (270000); 6 times the product of 300 and of 60 (for 57600 is added in twice), together with three times the square of 60.

$$
\begin{array}{rr}
(c) & 360 \\
 & \underline{3} \\
(3 \times 360) + 4 = & 1084 \\
 & \underline{4} \\
(3 \times 360 \times 4) + 4^2 = & 4336
\end{array}
$$

Questions on Involution and Evolution.

Define the terms power, root, involution, evolution, square, cube, exponent. Give a reason why the product of two equal numbers should be called a square, and that of three such numbers a cube. When are quantities rational? when irrational? Why are these terms applied? What is a surd?

On what fundamental axiom of arithmetic does Involution rest? Enunciate in order the propositions relating to the square of a number consisting of two parts. Give examples. Suppose a number consist of 1, of 4, of 12 parts, describe how its square is formed. By how much does the square of a number differ from the square of the number next above it? How is the square of a number related to the squares of its factors?

How many digits will be contained in the square of a number containing two digits? Why? What use is made of this fact in extracting the square root? What is the product of the sum and difference of two numbers? What is the square of their difference? By how much do the square of the sum, the sum of the squares, and the square of the difference of two numbers differ from one another? How is the square of half a number related to that of the number itself? How is the square of half the sum of two numbers related to their product?

How is the root of a fraction to be found? What is the most convenient form in which to extract such a root? Explain the reason of each of the following portions of the ordinary rule—pointing, making an even number of decimal places, doubling the part of the root, adding a new part, and multiplying by the new part. How may a root be obtained approximately? Suppose accuracy be required as far as $\frac{1}{n}$, how may it be obtained? By what tests may you ascertain by inspection whether a number is a perfect square?

Describe the manner in which the third power of a number is made up. How do the cubes of any two consecutive numbers differ? In the extraction of the cube root, why should we point every third figure? Give a reason for adding ciphers in the case of decimal fractions; and for tripling the square of the first found part. Take the number 178614, extract its square and also its cube root, analyzing the process in each sum, so as to show the separate meaning of every line.

Express in words the truths embodied in the following formulæ:—

$$(a+b)^2 = (a+b)\,a + (a+b)\,b = a^2 + 2ab + b^2$$

$$(a+b+c+d+e)^2 = a^2 + 2a\,(b+c+d+e) + b^2 + 2b\,(c+d+e) + c^2 + 2c\,(d+e) + d^2 + 2de + e^2$$

$$(a+b)(a-b) = a^2 - b^2, \text{ and } (a-b)^2 = a^2 - 2ab + b^2$$

$$\left(\frac{a+b}{2}\right)^2 - ab = \left(\frac{a-b}{2}\right)^2, \quad (a+1)^2 - a^2 = a + (a+1)$$

$$abc^2 = a^2b^2c^2, \quad \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}, \text{ and } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$abcd^3 = a^3b^3c^3d^3, \text{ and } \sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}$$

Example.—Find the cube root of 4822855.

$$4\dot{8}228\dot{5}50\dot{}(364$$
$$27$$
$$21228$$

1st Trial divisor $3 \times 30^2 = 2700$
(*a*). $(3 \times 30 \times 6) + 6^2 = 576$
$3276 \times 6 = 19656$
36 — 1572550
2nd Trial divisor (*b*). $3 \times 360^2 = 388800$
(*c*). $(3 \times 360 \times 4) + 4^2 = 4336$
$393136 \times 4 = 1572544$
6

Exercise CXXVIII.

Find the cube roots of the following numbers :—

1. 32768; 1157·625; 247673152.
2. 32461759; 78610563187; 5·088448.
3. 1·29503; 4173·281; 135796·744.
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9. Find the cube root of $\frac{5}{7}$ true to $\frac{1}{18}$, and that of $31\frac{2}{3}$ true to $\frac{1}{23}$.
10. Find $\sqrt{\frac{4}{9}}$ true to $\frac{1}{30}$, and $\sqrt{\frac{11}{13}}$ true to $\frac{1}{45}$.
11. Find $\sqrt{24\frac{1}{7}}$ true to $\frac{1}{13}$, and $\sqrt{273\frac{1}{4}}$ to $\frac{1}{32}$.
12. Find the approximate cube root of the following numbers in a decimal form :—
    (a). 729863; 527410; 6283·0542.
    (b). $\frac{3}{8}$; ·68; ·0654; 27·0298.
    (c). $17\cdot\dot{4}$; $18\cdot\dot{0}3\dot{6}$; $509\cdot\dot{7}$; $62\cdot87\dot{5}$.
13. The product of 3 equal numbers is 3189506048, what are they?

(*a*)
$$\begin{array}{r} 30 \\ 3 \\ \hline (3 \times 30) + 6 = 90 \\ 6 \\ \hline (3 \times 30 \times 6) + 6^2 = 576 \end{array}$$

(*b*). This number, 388800, is clearly 3 times the square of 360, for it contains 3 times the square of 300 (270000); 6 times the product of 300 and of 60 (for 57600 is added in twice), together with three times the square of 60.

(*c*)
$$\begin{array}{r} 360 \\ 3 \\ \hline (3 \times 360) + 4 = 1084 \\ 4 \\ \hline (3 \times 360 \times 4) + 4^2 = 4336 \end{array}$$

Questions on Involution and Evolution.

Define the terms power, root, involution, evolution, square, cube, exponent. Give a reason why the product of two equal numbers should be called a square, and that of three such numbers a cube. When are quantities rational? when irrational? Why are these terms applied? What is a surd?

On what fundamental axiom of arithmetic does Involution rest? Enunciate in order the propositions relating to the square of a number consisting of two parts. Give examples. Suppose a number consist of 1, of 4, of 12 parts, describe how its square is formed. By how much does the square of a number differ from the square of the number next above it? How is the square of a number related to the squares of its factors?

How many digits will be contained in the square of a number containing two digits? Why? What use is made of this fact in extracting the square root? What is the product of the sum and difference of two numbers? What is the square of their difference? By how much do the square of the sum, the sum of the squares, and the square of the difference of two numbers differ from one another? How is the square of half a number related to that of the number itself? How is the square of half the sum of two numbers related to their product?

How is the root of a fraction to be found? What is the most convenient form in which to extract such a root? Explain the reason of each of the following portions of the ordinary rule—pointing, making an even number of decimal places, doubling the part of the root, adding a new part, and multiplying by the new part. How may a root be obtained approximately? Suppose accuracy be required as far as $\frac{1}{n}$, how may it be obtained? By what tests may you ascertain by inspection whether a number is a perfect square?

Describe the manner in which the third power of a number is made up. How do the cubes of any two consecutive numbers differ? In the extraction of the cube root, why should we point every third figure? Give a reason for adding ciphers in the case of decimal fractions; and for tripling the square of the first found part. Take the number 178614, extract its square and also its cube root, analyzing the process in each sum, so as to show the separate meaning of every line.

Express in words the truths embodied in the following formulæ:—

$$(a + b)^2 = (a + b)\,a + (a + b)\,b = a^2 + 2ab + b^2$$

$$(a + b + c + d + e)^2 = a^2 + 2a\,(b + c + d + e) + b^2 + 2b\,(c + d + e) + c^2 + 2c\,(d + e) + d^2 + 2de + e^2$$

$$(a + b)\,(a - b) = a^2 - b^2, \text{ and } (a - b)^2 = a^2 - 2ab + b^2$$

$$\left(\frac{a + b}{2}\right)^2 - ab = \left(\frac{a - b}{2}\right)^2, \quad (a + 1)^2 - a^2 = a + (a + 1)$$

$$abc^2 = a^2b^2c^2, \quad \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}, \text{ and } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$abcd^3 = a^3b^3c^3d^3, \text{ and } \sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}$$

APPLICATION OF ARITHMETIC TO GEOMETRICAL MEASUREMENTS.

478. Many of the preceding rules are employed in the measurement of lines, surfaces, and solids.

Geometry investigates the proportions subsisting among the lines by which regular plane and solid figures are contained, and shows how those lines determine the magnitude of the figures themselves.

479. Every practical rule of mensuration is based upon some theorem of geometry, and arithmetic can only be applied to this purpose when we take for granted the truth of some such theorem. The following are among the most important of these, with their practical applications:—

480. *I. Every parallelogram, whatever be its form, is equal in area to a rectangle having an equal base and perpendicular altitude to itself.* (Euclid i., 36.)

The area of every triangle is half of that of a parallelogram having the same base and altitude as itself. (Euclid i., 41.)

Parallelograms are to one another in a ratio compounded of their bases and altitudes. (Euclid, Book vi., 1, *corollary*.)

481. Hence, To FIND THE AREA OF A PARALLELOGRAM—

Multiply the number of linear units in the base by the number of linear units in the altitude.

And, To FIND THE AREA OF A TRIANGLE—

Find the area of a parallelogram having the same base and altitude, and divide the result by 2.

482. The foot is generally taken as the standard of linear measure by artizans, and the square foot is the principal superficial unit. Each of these units is subdivided into twelfths, twelfths of twelfths, &c. Hence the ordinary method of making such calculations is called *duodecimal*.* The two subdivisions are as follows:—

1 foot = 12 *primes* or inches = 144 seconds = 1728 thirds, &c.
1 sq. foot = 12 primes = 144 *seconds* or inches = 1728 thirds, &c.

* From Latin *duodecim*, twelve, or 2 (*duo*) + 10 (*decem*).

The *prime*, or twelfth of a *linear* foot, is a *linear* inch, but the *second*, or 144th of a *square* foot, is a square or superficial inch, the prime or twelfth of a square foot containing 12 square inches.

483. It is usual to express these subdivisions by accents, thus:—

$$7 \text{ feet } 6'\ 3''\ 2'''\ 8'''' = 7 + \frac{6}{12} + \frac{3}{12^2} + \frac{2}{12^3} + \frac{8}{12^4} \text{ feet.}$$

Now because a rectangular area 1 foot long and 1 foot wide is a square foot—

∴ *Feet multiplied by feet give square feet.*

And because such an area 1 foot long and 1 inch wide contains 12 square inches—

∴ *Feet multiplied by inches give superficial primes.*

And because such an area 1 inch long and 1 wide is a square inch—

∴ *Inches multiplied by inches give superficial seconds.*

And because such an area 1 inch long and 1 second wide contains one-twelfth of a square inch—

∴ *Inches multiplied by seconds give superficial thirds.*

And in the same manner it may be seen that—

Feet multiplied by seconds give superficial seconds or square inches.

Seconds multiplied by seconds, or } *give superficial fourths.*
Inches multiplied by thirds ...... }

Example I.—Find the area of a floor 17 feet 2′ 6″ by 11 feet 8′ 9″.

ft.	in.	sec.			
17	2	6			
11	8	9			
189	3	6			$= 189 + \frac{3}{12} + \frac{6}{144} = (17\ 2'\ 6'') \times 11$
11	5	8	0		$= 11 + \frac{5}{12} + \frac{8}{144} = (17\ 2'\ 6'') \times 8'$
1	0	10	10‴	6⁗	$= 1 + \frac{10}{144} + \frac{10}{1728} + \frac{6}{20736} = (17\ 2'\ 6'') \times 9''$
211	10′	0″	10‴	6⁗	

Example II.—What surface is contained by a triangle whose base is 18 feet 9 inches, and perpendicular height 14 feet 7 inches?

	ft.	in.			
	18	9			
	14	7			
	262	6′			$= (18\ 9') \times 14$
	10	11′	3″		$= (18\ 9') \times 7'$
	2)273	5′	3″		= area of rectangle having the same [base and altitude
Square feet,	136	8′	7″	6‴	= area of triangle.

484. When the fractions of a foot, given in the question, can be easily expressed in decimals, the area can be found more readily than by this method.

Example III.—Find the area of a surface measuring 7 feet 6 inches by 11 feet 9 inches.

Duodecimal method.

ft.	in.		
7	6		
11	9		
82	6		= (7 6′) × 11
5	7	6″	= (7 6′) × 9′
88	1′	6″	= $88\frac{18}{144}$ feet.

Decimal method.

$$\begin{array}{r} 11{\cdot}75 \\ 7{\cdot}5 \\ \hline 5875 \\ 8225 \\ \hline 88{\cdot}125 \end{array} = 88\tfrac{125}{1000} \text{ feet.}$$

485. As cost of paving, flooring, or tiling is usually calculated at a certain rate per yard or foot, it is generally convenient, when the question refers to price, to reduce the whole of the dimensions to one denomination, and represent them all as fractions of a foot.

Example IV.—What would be the cost of carpeting a room 32 feet 6 inches long and 23 feet 9 inches wide, at 5s. 6d. per square yard?

I. By Ordinary Fractions.

$32\frac{1}{2} \times 23\frac{3}{4}$ feet
$= (\frac{65}{2} \times \frac{95}{4})$ square feet
$= 771\frac{7}{8} \times \frac{1}{9} =$ square yards
$= 85\frac{55}{72}$ square yards

$(85\frac{55}{72} \times 5\frac{1}{2})$s. = cost of carpeting
= £23 11s. $8\frac{1}{4}$d.

II. Duodecimally.

	ft.	in.	sec.
	32	6	
	23	9	
	747	6	
	24	4	6
9)	771	10	6
Square yards,	85	9	8

	£	s.	d.
85 at 5s. 6d. =	£23	7	6
$\frac{9}{12}$ .... =	0	4	$1\frac{1}{2}$
$\frac{3}{144}$ .... =	0	0	$0\frac{3}{4}$
	£23	11	$8\frac{1}{4}$

EXERCISE CXXIX.

1. Find the area of a rectangle 2 ft. 6′ 8″ by 17 ft. 8′ 10″, and of one 3 ft. 4′ 7″ by 1 ft. 9′ 11″.

2. Find the area of a rectangle 36 ft. 7′ 11″ by 9 ft. 8′ 3″, and of one 20 ft. 8′ by 15 ft. 0′ 10″.

3. Find the area of a rectangle 111 ft. 1′ 5″ by 14 ft. 7′, and of one 10 ft. 8′ 7″ 6‴ by 35 ft. 8′ 10″.

4. Find the area of a rectangle 7 ft. 6′ 9″ by 13 ft. 8′ 14″ 7‴, and of one 20·72 feet by 18·71 inches.

5. Find the area of a rectangle 17·25 feet by 6·231 yards, and of one 238·19 feet by 61·518 feet.

6. Find the area of a rectangle $72\frac{1}{4}$ feet by $56\frac{3}{4}$ feet, and of one $86\frac{1}{12}$ feet by $75\frac{3}{8}$ feet.

7. What is the area of a triangle whose base is 14·672 feet and perpendicular height 11·98 feet?

8. What is the area of a triangle whose base is 3·08 feet and perpendicular height 6·589 feet?

9. What is the area of a triangle whose base is 5·27 yards and perpendicular height 3·896?

10. What is the area of a triangle whose base is 37 ft. 6′ 11″ and perpendicular height 17 ft. 8′ 3″?

11. What is the length of a room containing 47 square yards of flooring and whose breadth is 18 ft. 5′?

12. How wide is a rectangle containing 156 ft. 8′ if its length be 17 ft. 9′ 4″?

13. How wide is a room containing 392 square ft. 5′ 8″ if its length be 23 ft. 8 in.

14. An area is $250\frac{5}{8}$ feet, its length $17\frac{1}{4}$ feet; find the breadth.

15. What length of carpet $\frac{3}{4}$ of a yard wide will cover a floor $6\frac{1}{4}$ yards long by $5\frac{1}{4}$ wide?

16. 69 yards of carpet $\frac{3}{4}$ of a yard wide cover a room $10\frac{1}{2}$ yards long; how wide is the room?

17. How much cloth will cover a room whose length is 12 feet 6 inches and breadth 2 feet 9 inches, and what would it cost at 5s. 6d. per square yard?

18. How many square feet in a floor whose length is $10\frac{3}{8}$ yards and breadth $5\frac{1}{2}$, the price of paving being 2s. per square foot?

19. How many square feet of paper will cover the walls of a room 20 ft. 10 inches long, 16 feet broad, and 10 feet 8 inches high?

20. What will remain out of 393 square feet of carpeting after covering a floor 23 feet 8 inches by 16 feet 7 inches?

21. Find the cost of covering a room $23\frac{3}{4}$ feet long by $16\frac{1}{4}$ wide, at 2s. 9d. per square yard.

22. What is the cost of carpeting a room $16\frac{1}{2}$ feet square, at 4s. $10\frac{1}{2}$d. per yard, if the carpeting be 2 feet wide?

23. What length of carpet $\frac{3}{4}$ yard wide will cover a room 42 feet 5 inches long by $31\frac{3}{4}$ wide?

24. Compare the magnitudes of two triangles, the one having a base $17\frac{1}{2}$ and an altitude $13\frac{3}{4}$, and the other having a base $20\frac{1}{3}$ and an altitude $11\frac{5}{8}$.

25. Compare the magnitude of a parallelogram 25 ft. 6′ by 17 ft. 7′ with that of a square on a line 23·67 feet long.

26. Compare the magnitudes of two rectangles, one of which is 17 ft. 8′ 7″ by 23 ft. 5′, and the other 11 ft. 6′ by 27 ft. 7′ 11″.

27. Find the difference between 27 sq. feet and 27 feet square.

28. Find the price of a rectangular piece of ground 52 feet 8 inches by 34 feet 9 inches, at 25s. per square foot.

29. How broad is a yard 36 feet 6 inches long, when the cost of paving it is £12 6s. $6\frac{1}{4}$d., at 5s. 3d. per square yard?

30. Find the cost of paving a yard 3 feet 10 inches broad and 12 feet 11 inches long, at 1s. 1d. per square foot.

486. *II. The square described on the greatest side of a right angled triangle is equal to the sum of the squares described upon the other two.* (Euclid i., 47.)

To FIND A SIDE OF A RIGHT ANGLED TRIANGLE—

487. The number representing the linear units in the hypotheneuse equals the square root of the sum of the second powers of the number of linear units in the other two sides.

EXERCISE CXXX.

1. Find the hypotheneuse of a right angled triangle whose sides are 15 and 7, and of another whose sides are 9 and 16.

2. The two sides of a right angled triangle are 120·5 and 83; by how much does the hypotheneuse exceed the greater of those sides?

3. If the hypotheneuse be 26, and one side 9, what is the other? If the hypotheneuse be 153, and the base 64, what is the altitude?

4. The foot of a ladder 30 feet long is 14 feet from a house, and its top reaches the upper part of a window; when it is drawn away to a distance of 17 feet, the top reaches the lower edge of the window; how high is the window?

5. How long should a rod be which joins the extremities of two walls, of which one is 8 feet and the other 6 feet, if the two walls meet at a right angle?

6. Find the diagonals of 3 squares whose sides respectively are 14, 20·6, and 33 feet in length.

7. What are the sides of three squares whose diagonals are respectively 25, 57·6, and 85·4 feet in length?

8. By how much does the square described on a line 5·73 inches long exceed one on a line of half the length?

9. What is the length of the side of a square piece of ground equal in area to a rectangle containing 4970·25 square yards?

10. If a rectangle contains 14723 square yards, and one side measures 506 yards, what is the length of the diagonal?

11. By how much does the square described upon the diagonal of a square exceed the square itself?

12. A rectangle has for one side a line 25 feet, and for another a line 9·9 yards long. Compare the magnitudes of these squares described on the two sides, and the diagonals respectively.

13. If two vessels sail from the same port, the one $17\frac{1}{2}$ leagues due east, and the other 41·6 leagues due north, how far will they be apart?

14. If a ladder 80 feet long be so placed as to reach a window 40 feet high on one side of the street, and 30 feet high on the other, how wide is the street.

15. What are the sides of two squares whose areas are 5083·6 inches and 17·8 feet respectively.

488. *III. The ratio of the diameter to the circumference of every circle is nearly that of* 1 *to* 3·14159.*

* This may be proved in two ways:—1. As a deduction from the formulæ in analytical trigonometry, which give a numerical equivalent in terms of the radius, for the sine of an angle so small that the arc does not appreciably differ from the sine itself: and, 2, By actual measurement. On unrolling the circumference of a circle, or finding a straight line of the same length, it is found impossible to express *precisely* the relation between this line and the diameter. But if the diameter contain 10 parts the circumference is observed to contain rather more than 31 such parts: if the diameter be 100, the circumference rather exceeds 314; if the diameter be 1000 the circumference is 3141; if the diameter has 100000 parts the circumference has 314159, &c.; each is a nearer approximation to the truth than its predecessor,

489. TO FIND EITHER THE CIRCUMFERENCE OR DIAMETER OF A CIRCLE WHEN THE OTHER IS KNOWN—

Multiply the diameter by 3·14159 to find the circumference, or divide the circumference by 3·14159 to find the diameter.

EXERCISE CXXXI.

1. Find the circumferences of circles whose diameters measure 17, 53·6, 247, and 10·9 feet respectively.

2. Find the diameters of circles whose circumferences measure 154, 208·6, and 4058 feet respectively.

3. The length of $\frac{1}{360}$ of the earth's circumference is about $69\frac{1}{23}$ miles; what is the earth's diameter?

4. What is the diameter of a circle whose semi-circumference is 54 yards?

5. Find the length of the French unit of length (a *metre*), which is a ten-millionth part of the distance from the equator to the pole, on the supposition that the diameter of the earth is 7912 miles.

6. How many degrees are there in an arc whose length is 73·5, radius 115·6?

7. Find the radius of a circle whose circumference is equal to $\frac{1}{3}$ of a mile.

8. Find the length of the quadrant of a circle, one-fourth of whose radius is 9·75 yards.

9. Find the number of degrees in an arc, whose length is $\frac{1}{8}$ the circumference, when $\frac{3}{4}$ the radius is 76·25 feet.

10. If the semi-circumference of a circle, whose radius is 98·5 feet, is equal in length to the quadrant of another, what is the diameter of this last?

11. What is the length of the arc of a circle whose radius is 24 ft. 6 in., and which contains the same number of degrees as the arc of another circle whose length is 5 ft. 4 in., to radius 9·12 ft.?

12. What is the number of degrees in the arc of a circle whose diameter is 79 ft. 10 in., and which is equal in length to the arc of another circle containing 26° 4′ 13″, to a diameter of 98 ft. 8 in.?

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but the two lines are incommensurable however far the comparison be carried. The following number expresses the truth as far as to 35 decimal places:—

3·14159265358979323846264338327950288.

490. *IV. The areas of circles are to one another as the squares of their diameters.*

*The area of every circle is equal to that of a rectangle whose base is the radius and whose altitude is equal to one-half of the length of the circumference; or it is equal to that of a rectangle whose base is the diameter, and whose altitude is one-fourth of the circumference.**

491. TO FIND THE AREA OF A CIRCLE—

Multiply the square of the radius by 3·14159, or multiply the square of the diameter by ·7854.†

EXERCISE CXXXII.

1. Find the areas of three circles whose radii are 7, 8, and 9 respectively.

2. Find the areas of three circles whose diameters are 75, 80·5, and 7·19 respectively.

3. If the radius of a circle be 3·5 feet, find the side of a square which shall have the same area.

4. What is the radius of a circular field containing 2½ acres?

5. Compare the magnitudes of three circular plots of ground whose radii are 17, 18, and 19 respectively.

6. The diameters of two concentric circles are 173 and 191·6 feet; find the space included between the two circumferences.

7. What is the area of a gravel walk round a circular grass plot whose radius is 23 ft., the width of the walk being 5 ft. 6 in.?

8. What is the area of a path round a circular flower bed whose diameter is 11·72 feet, the width of the path being 3 ft. 8 in.?

9. The area of the space occupied by a circular tower is 687 yards, the area of a moat round it is 1280 yards; what is the width of the moat in feet?

10. A man wishing to find the number of acres covered by a circular pond, walked round it at the rate of 3½ miles an hour, and found that it took him 2½ hours to complete the journey; required the area of the pond.

* These propositions are readily deducible from Euclid, Book vi., prop. 20.

† ·7854 = 3·14159 ÷ 4.

11. How many acres are there in a circular field, the cost of planting a hedge round it being £72 10s., at 2s. 4d. per yard?

12. I have purchased a circular field for £690 12s. 6d., at £65 per acre; what will be the cost of digging a ditch round it at 5d. per yard, and of dividing it into two equal parts by a wall which will cost 4s. 9d. per yard in building?

13. Twelve persons can sit round a circular table, allowing 22 inches for each; what will be the price of a cloth to cover it, and extend 1 foot over the edge all round, at 2s. 6d. per square yard?

492. *V. The product of the numbers representing the three dimensions of a rectangular solid, or parallelopiped, represents the number of cubic units in the solid itself.**

493. TO FIND THE CUBIC CONTENTS OF A PARALLELOPIPED—

Find the continued product of the numbers representing the length, breadth, and thickness of the solid.

494. It is usual to employ the duodecimal method described in (482) to express the solid contents; a cubic foot being divided into *primes*, *seconds*, *thirds*, &c. Only here it must be remembered that whereas a lineal prime, or $\frac{1}{12}$ of a foot, is a lineal inch; a superficial second, or $\frac{1}{144}$, is a superficial inch; so a cubical third, or $\frac{1}{1728}$, is a cubic inch.

Example.—How many cubic feet and inches are there in a solid whose breadth is 9 ft. 3 in., length 11 ft. 5 in., and height 3 ft. 2 in.?

	ft.	in.			
	9	3			
	11	5			
	101	9			
	3	10	3		
square feet	105	7′	3″		= Superficial content of one side of the solid
	3	2			
	316	9	9		
	17	7	2	6	
cubic feet	334	4′	11″	6‴	= Solid content of the whole figure
Reducing the primes & seconds to thirds or cubical inches.		12			
		59″			
		12			
		714‴			Ans. 334 cubic ft. 714 cubic in.

* See note ante, p. 235.

Exercise CXXXIII.

1. What are the solid contents of a rectangular mass, 15 ft. 6 in. long, 18 ft. 5 in. wide, and 23·5 ft. thick?

2. Find the content of a block of stone 17 ft. 9 in. long, 14 ft. 3 in. broad, and 5 ft. 6 in. thick; and its price at 4d. per cubic foot.

3. The weight of the cubic foot of water is about 1000 ounces; what weight of water will fill a cistern 4 ft. 6 in. long, 3 ft. broad, and 4 ft. 3 in. deep?

4. What will be the cost of a marble block 37 ft. 8 in. long, 8 ft. broad, and 6 ft. 5 in. thick, at 5s. 6d. per solid foot?

5. If a brick be 9 in. long, 4 in. wide, and 3 in. thick, how many will be required for a wall 1 foot 10 in. thick, 100 yards long, and $4\frac{1}{2}$ yards high?

6. The weight of a cubic foot of Portland stone is 156 lbs.; find the weight of a block 7 ft. long, 3 ft. 9 in. broad, and 2 ft. 1 in. thick.

7. The weight of a cubic foot of oak is 58 lbs.; what is the weight of 3 beams, each being 12 ft. 6 in. long, 2 ft. 3 in. broad, and 1 ft. 6 in. thick?

8. Gold sells at £3 17s. 6d. per oz.; what is the value of a bar 6 in. long, and $1\frac{1}{2}$ in. in breadth and thickness, a cubic inch weighing 131 oz.?

9. What is the cost of a marble slab, 6 ft. 3 in. long, 2 ft. 8 in. broad, 4 in. thick, at 14s. 6d. per cubic foot? What is the weight of the slab, one cubic foot weighing 170 lbs.?

495. *VI. Spheres are to one another as the cubes of their diameters. The cubic content of a cylinder equals that of a rectangular solid having the same altitude, and whose base is equal to the area of the circle.*

The cubic content of a sphere is two-thirds of a cylinder having the same diameter and altitude.

*The cubic content of a cone is one-third of a cylinder having the same diameter and altitude.**

* These propositions are all demonstrated in any ordinary treatise on solid geometry; we do not quote the number of the propositions as they stand in Euclid because that part of his "Elements" which refers to this subject is seldom used.

496. TO FIND THE CONTENT OF A PARALLELOPIPED, A PRISM, OR A CYLINDER—

Multiply the number representing the area of the base by the number of linear units in the altitude.

497. TO FIND THE CONTENT OF A CONE—

Multiply the area of the base by one-third of the altitude.

498. TO FIND THE SOLIDITY OF A SPHERE—

* Multiply the cube of the radius by 4·18879.

EXERCISE CXXXIV.

1. How many cubic inches are there in two spheres whose diameters are 17·725 ft. and 2·1 feet respectively?

2. The circumference of a sphere is 125½ yards; find its solidity, also that of another sphere whose circumference is 24900 miles.

3. The radius of the base of a cylinder is 56 ft. 8 in., and the height is half the circumference; required the solidity.

4. The length of a cylinder is 127½ ft., and the radius 13¾ ft.; what is the solidity?

5. What is the solidity of a cylinder whose length is 72·25 ft., and the circumference ¼ the length?

6. Required the cubic contents of a cylinder, having a length of 174·2 ft., and the circumference 17·42 ft.

7. The length of a hollow roller is 4 ft., exterior diameter 2 ft., and the thickness of the metal ¾ of an inch; determine its solidity.

8. If a cylinder, whose lengthi s 13 ft. 4 in., contains 1728 cubic feet, what length must be cut off, so that it may contain ⅕ that number of cubic feet?

9. If a cylinder contains 692·5 cubic feet, its length being 74·25 ft., how long must it be to contain 138·5 cubic feet?

10. The diameter of a cone is 17 ft. 9 in., the perpendicular height twice the circumference; required the solidity.

11. Required the solidity of a cone whose perpendicular height is 25 ft. 11 in., and the diameter 2 ft. 7 in.

12. The circumference of a cone is 25 ft., and the height four times the radius; required the solidity.

* *4·18879* = ⅔ of 6·28318 which is the ratio of circumference to radius.

PROGRESSION AND LOGARITHMS.

499. Any set or series of numbers of which each is related to its successor according to some fixed law is said to be in Progression.

SECTION I.—PROGRESSION BY EQUAL DIFFERENCE, OR ARITHMETICAL PROGRESSION.

500. When any set or series of numbers is so arranged that each differs from that which precedes it by the same number, they are said to form an *Equi-different* or *Arithmetical Progression.*

The number by which they differ is called the Common Difference.

Let there be two sets of numbers—

I.	5	8	11	14	17	20	23	26	29	· · ·
II.	55	50	45	40	35	30	25	20	10	· · ·

In (I.) each number exceeds that which is before it by the constant difference 3. In (II.) each number is less than that which precedes it by the common difference 5. The former is an example of *ascending* and the other of *descending* progression.

501. *In an ascending progression the second term equals the first, plus the common difference; the third equals the first, plus twice the common difference; and generally, any term in the series is made up of the first term, plus the common difference multiplied by the number of terms before it.*

Observation.—If the word *minus* be substituted for *plus* throughout this proposition, it will obviously be true for all cases of descending progression.

502. *Demonstrative Example.*—If 3 be the first term, and 4 the common difference, then—

The second term $= 3 + 4$
The fifth term $= 3 + (4 \times 4)$
The ninth term $= 3 + (4 \times 8)$
The hundredth term $= 3 + (4 \times 99)$.

503. Suppose a, e, i, o, u are in ascending progression, and that d be their common difference,

$$\begin{aligned} \text{Then } e &= a + d \\ i &= e + d = a + d + d = a + 2d \\ o &= i + d = a + 2d + d = a + 3d \\ u &= o + d = a + 3d + d = a + 4d. \end{aligned}$$

If n be the number which denotes the place of a term, then $(n - 1)$ denotes how many terms stand before it.

General Formula.—Let a = first term, and d = common difference; nth term $= a + (n - 1)\,d$.

If the progression be descending, then we should have—

$$\begin{aligned} e &= a - d \\ i &= e - d = a - d - d = a - 2d \\ o &= i - d = a - 2d - d = a - 3d \end{aligned}$$

and hence nth term $= a - (n - 1)\,d$.

504. TO FIND ANY GIVEN TERM OF AN ARITHMETICAL PROGRESSION—

RULE.

Multiply the common difference by the number of terms *minus one*, add this product to the first term if the progression be ascending, and subtract it from the first term if the progression be descending.

EXERCISE CXXXV.

Find what the following terms are:—

Example.—Find the 21st term of a progression whose first term is 5, and whose common difference is 8.

Here 21st term $= 5 + (20 \times 8) = 165$.

1. First term 7, common difference 8; find the 9th term.
2. First term 6, common difference 12; find the 20th term.
3. First term 7, common difference 3; find the 12th term.
4. First term 3, common difference 2; find the 24th term.
5. First term 5, common difference 4; find the 100th term.
6. First term 6, common difference 8; find the 23rd term.
7. First term 11, common difference 2; find the 15th term.
8. First term 14, common difference 3; find the 23rd term.

9. First term 7, common difference 9; find the 11th term.
10. First term 1, common difference $11\frac{1}{2}$; find the 5th term.
11. First term 20, second 17; find the 5th term.
12. First term 100, second 94; find the 7th term.
13. First term 53, second 51; find the 20th term.
14. First term 33, second $32\frac{1}{2}$; find the 14th term.
15. First term 45, second 41·25; find the 6th term.

505. A method is easily deduced from this by which we may insert any number of terms between two numbers, so that the whole shall form an equi-different series. Terms thus inserted are called *differential or arithmetical means.*

For since the last term contains the first + the product of the common difference into the number of terms minus one,

506. TO INSERT ANY NUMBER OF ARITHMETICAL OR DIFFERENTIAL MEANS BETWEEN TWO NUMBERS—

RULE.

Subtract the less from the greater, and divide the remainder by the total number of the terms required to be inserted, plus one. This will give the common difference.

Example.—Insert 9 differential means between 3 and 53.

Here, because there are in all 11 terms, 53 or the last term consists of the first term, plus 10 times the common difference.

$$\therefore \frac{53-3}{10} = 5 = \text{the common difference.}$$

$\therefore 3 + 5 =$ 2nd term, $3 + \overline{2 \times 5} =$ the 3rd term, and the other terms proceed in the same order.

EXERCISE CXXXVI.

1. Insert 15 differential means between 5 and 37.
2. Insert 3 differential means between 8 and 36.
3. Insert 16 differential means between 5 and 56.
4. Insert 8 differential means between 6 and 600.

507. *If in a progression any two numbers be taken, equally distant from a third, their sum equals twice the third.*

This is shown in (54), for if in the following series—

6, 11, 16, 21, 26, 31, 36, 41, 46,

any one number, as 21, be taken, it is evident that this added to itself will equal the sum of 16 and 26, or of 11 and 31, or of 6 and 36, each of these pairs consisting of two numbers equally distant from 21, and one of each pair being as much less as the other is greater than the central figure.

The number thus between the other two is their arithmetical mean or their average.

508. *Observation.*—The term *average* is also applied in cases when three, four, or many numbers are concerned. Thus, if there be 3 unequal numbers and we add them together and take one-third of the sum, we find their average. So if there be 7 unequal numbers, and we divide their sum by 7, we discover what 7 equal numbers would amount to the same sum, and any one of them is the average of the whole.

TO FIND THE ARITHMETICAL MEAN OR AVERAGE OF NUMBERS—

RULE.

Add them together, and divide their sum by the number of terms.

Example.—What is the arithmetical mean of 34 and 80?

Here $\frac{34 + 80}{2} = 57 =$ the arithmetical mean.

And 34, 57, and 80 form an arithmetical or equi-different progression.

EXERCISE CXXXVII.

(a). Find the arithmetical means of the following pairs of numbers:—

1. 72 and 36; 45 and 69; 83 and 57; 111 and 127.
2. 38 and 64; 25 and 135; 41 and 93; 61 and 85.
3. 100·2 and 7·8; 6·5 and 114·5; 8·4 and 11·26.
4. $\frac{3}{4}$ and 1·25; $\frac{7}{9}$ and $\frac{11}{10}$; $\frac{18}{25}$ and $\frac{6}{31}$.
5. $4\frac{1}{8}$ and $6\frac{2}{3}$; $27\frac{1}{9}$ and $35\frac{1}{12}$; 18·6 and 35·4.

(b). Find the average in the following sets of numbers:—

1. 18, 19, and 20; 15, 18, and 21; 23, 27, and 32.
2. 4, 5, 6, and 7; 8, 9, 10, and 11; 17, 23, 25, and 26.
3. 14, 25, 8, and 12; 4, 11, 13, and 8; 25, 16, 33, and 44.
4. 79, 86, 53, and 12·5; 7·9, 9·9, and 8·9; 6·25, 3·1, and 4·7.

509. *In every arithmetical progression, the sum of any two terms at equal distance from the two extremes, equals the sum of the two extremes.*

Demonstrative Example I.—

2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35.

Here it follows from (55) that $2 + 35 = 5 + 32 = 8 + 29 = 11 + 26 = 14 + 23 = 17 + 20$.

General Formula.—If a, b, c, d, e, f, g be in arithmetical progression,

Then $a + g = b + f = c + e = 2d$.

Thus (55) every such series may be resolved into a number of pairs or couples, each having the same sum. And since the number of such pairs must always be half the number of terms, it follows that the sum of any two which form a pair, repeated as many times as there are pairs in the series, will give the sum of all the numbers in the series.

Demonstrative Example II.—

Let the series be 3, 7, 11, 15 · · · · · 35, 39, 43, 47,
Then placing these numbers
in the reverse order, 47, 43, 39, 35 · · · · · 15, 11, 7, 3.

It is evident that the sum of every pair in the vertical lines is the same. Let the progression be of any length, let the unknown sum be denoted by S, and let n represent the number of terms.

a, b, c · · · · · x, y, z

z, y, x · · · · · c, b, a

If now we write the same set of terms in the reverse order, the sum of the two series will be 2 S. But because (509) $a + z = b + y = c + x$ · · · · · it follows that—

$2\,S = a + z$ taken as many times as there are terms.

Or $2\,S = (a + z)\,n$.

And if we divide both by 2, then $S = \dfrac{(a + z)\,n}{2}$.

General Formula.—If $a =$ first term, $z =$ the last term, and n the number of terms;

Then Sum of n terms $= (a + z) \times \dfrac{n}{2}$.

But since (502) $z = a + \overline{n - 1}\,d$ when d is the difference,

Sum of n terms $= (a + a + \overline{n - 1}\,d)\,\dfrac{n}{2} = (2a + \overline{n - 1}\,d)\,\dfrac{n}{2}$.

510. TO FIND THE SUM OF AN EQUI-DIFFERENT SERIES—

RULE.

Find the last term by (504). Add the first and last terms together, and multiply by half the number of terms.

511. *Observation.*—Dividing the sum of the first and last terms by 2 gives us the *average* of the whole series; and n times the average is evidently the same as the sum of n unequal numbers.

Example.—What is the sum of 25 terms of the progression 2, 7, 12?

Here by (501) the 25th term $= 2 + \overline{5 \times 24} = 122$.

Therefore the sum $= \dfrac{(2 + 122)\ 25}{2} = \dfrac{25 \times 124}{2} = 1550$.

EXERCISE CXXXVIII.

Sum the following series:—

1. 3, 9, 15, to 13 terms; and 1, 9, 17, to 100 terms.
2. 8, 15, 22, to 12 terms; and 1, 3, 5, to 25 terms.
3. 4, 7, 10, to 23 terms; and 12, 17, 22, to 86 terms.
4. 1, $1\frac{1}{2}$, 2, to 18 terms; and 5, $5\frac{1}{3}$, $5\frac{2}{3}$, to 21 terms.
5. 6, 6·25, 6·5, to 30 terms; and 11, 13·7, 15·4, to 15 terms.
6. 116, 108, 100, to 10 terms; and 270, 260, 250, to 11 terms.
7. $\frac{5}{7}$, 1, $1\frac{2}{7}$, to 8 terms; 6, $5\frac{1}{2}$, 5, to 25 terms.
8. 39, $35\frac{1}{2}$, 32, to 20 terms; and 1, $\frac{3}{2}$, 2, to 12 terms.
9. How many times does the hammer of a clock strike in a week?
10. What debt can be discharged in 2 years by monthly payments, at 6d. the first month and 8d. additional in each succeeding month?
11. A man travels 5 miles on one day, 8 the next, 11 the next, increasing his journey regularly at the same rate; in how many days will he travel 735 miles?
12. Add together all the numbers from 3 to 5000, both included.
13. A man receives a regular increase of 9d. a week in wages; if he begin at 10s. what will he earn a year and a half hence?
14. A falling body descends 16·1 feet in the first second, 48·3 in the second, and 80·5 in the third; how far would it fall in the seventeenth second of its descent?

512. *The sum of the series of odd numbers, beginning with unity, and carried to any given number of terms, always equals the square of that number.*

Demonstrative Example.—Let it be required to find the sum of 20 terms of the series 1, 3, 5, &c.

Here by (503) the 20th term $= 1 + \overline{19 \times 2} = 39$.

And by (510) $\text{Sum} = \dfrac{1 + 39}{2} \times 20$.

$$\text{But } \frac{1 + 39}{2} = 20. \quad \therefore \text{ Sum} = 20 \times 20 = 400.$$

In the same manner it might be shown that in this series the sum of 27 terms $= 27^2$; the sum of 100 terms will be 100^2, &c.*

General Formula.—If $a = 1$ and $d = 2$,

$$\text{Then Sum} = \frac{(2 + (n - 1)\, 2)\, n}{2} = \frac{(2 + 2n - 2)\, n}{2} = n^2.$$

513. *Corollary.—If there be any arithmetical series such that the common difference is double the first term, the sum of* n *terms of that series equals the product of the first term into* n^2.

Example.—16, 48, 80, 112, &c.

Here first term = 16 and common difference 32, or 2×16.

† Hence the sum of 7 terms $= 16 \times 7^2$.

EXERCISE CXXXIX.

1. Find the sums of 24, of 57, and of 93 terms of the series of odd numbers.
2. Find the sum of 25 terms of the series 3, 9, 15, &c.
3. Find the sum of 237 terms of the series 2, 6, 10, &c.
4. Find the sum of 185 terms of the series 5, 15, 25, &c.
5. Find the sum of 25 terms of the series 3·5, 10·5, 17·5, &c.
6. Find the sum of 17 terms of the series 12, 36, 60, &c.
7. How far will a body fall in 19 seconds?

$1^2 + 3 = 2^2$
$2^2 + 5 = 3^2$
$3^2 + 7 = 4^2$
$4^2 + 9 = 5^2$

* The student should carry this table further, and then compare this truth with that given in (429) and endeavour to detect the same principle under both forms.

† This progression represents the number of feet through which a falling body passes in successive seconds, and the rule is therefore of importance. To find the space traversed in n seconds, multiply 16 by n^2.

3. Find the 8th term of the series 2, 6, 18, &c.
4. Find the 5th term of the series 7, 28, 112, &c.
5. Find the 11th term of the series 6, 3, $\frac{3}{2}$, &c.
6. Find the 8th term of the series 5, 20, 80, &c.
7. Find the 7th term of the series 3, 30, 300, &c.
8. Find the 6th term of the series 2, $\frac{2}{3}$, $\frac{2}{9}$, &c.
9. Find the 12th term of the series 2, 6, 18, &c.
10. Find the 10th term of the series 12, 6, 3, &c.

518. *The square root of the product of two numbers is always their proportional mean.*

Demonstrative Example.—Let a proportional mean be required between 2 and 200.

Here by (324) if 2, x, and 200 be in proportion, $2 \times 200 = x^2$.

And because $2 \times 200 =$ the square of the mean,

$\therefore$ the mean $= \sqrt{2 \times 200} = \sqrt{400} = 20$.

$\therefore$ 2 : 20 : 200 are in geometrical progression, and 20 is the proportional mean.

General Formula.—$a : \sqrt{ab} : b$.

TO FIND A PROPORTIONAL MEAN BETWEEN TWO NUMBERS—

RULE.

519. Find their product and extract its square root.

EXERCISE CXLII.

Find a mean proportional between the following numbers:—

1. 18 and 162; 29 and 1421; 6 and 54.
2. 15 and 240; 18 and 450; 13 and 1573.
3. 2 and 18; 35 and 46; 279·2 and 3472·9.

520. To insert any number of proportional means between any two numbers it is only necessary to find the common ratio. But to do this often involves the extraction of higher roots. Thus, if the first term and the fourth are given, and we are required to find the second and third, we must seek the common ratio.

But since the 4th term = the first term × the cube of the common ratio, we must divide the fourth term by the first, and then extract the third root of the quotient in order to find a common ratio. Similarly, if we have to insert 7 proportional means between *two numbers*, these two numbers will form the 1st and 9th terms of

the series. But the 9th term is made up of the product of the first, and the 8th power of the common ratio. Here therefore it would be necessary to divide the greater number by the less, and to extract the 8th root of the quotient; this would give the common ratio.

General Formula.—If r be the common ratio and n the number of terms, a the first term and z the last,

$$\text{Then } r = \sqrt[n-1]{\frac{z}{a}}$$

521. It will thus be seen that the common ratio can be obtained by ordinary arithmetic, only when the number representing the root to be found is either 2 or 3.

522. *In every geometrical progression, the product of any two terms equally distant from the two extremes is the same as the product of the extremes.*

Demonstrative Example.—5 : 15 : 45 : 135 : 405 : 1215 : 3645.

Here because 15 is as many times more than 5 as 1215 is less than 3645 $\therefore$ (126) $15 \times 1215 = 5 \times 3645$.

And because 45 is as many times more than 5 as 405 is less than 3645 $\therefore$ (126) $45 \times 405 = 5 \times 3645$.

And because 135 is as many times more than 5 as it is less than 3645 $\therefore$ (124) $135^2 = 5 \times 3645$.

Hence $5 \times 3645 = 15 \times 1215 = 45 \times 405 = 135^2$.

General Formula.—If $a : b : c : d : e : f : g$,

Then $ag = bf = ce = d^2$.

Hence (127) *every geometric series may be resolved into a number of pairs of factors, each pair having the same product as the square of the middle term, or as the product of the two extremes.*

523. This truth does not, however, like the analogous truth in Arithmetical Progression (508), enable us to find the *sum* of the series;* it is necessary, therefore, to seek some other method of arriving at this result.

* If the inference were of any value, it would be interesting to notice that we might here obtain an estimate of the *product* of all the terms of a geometric series, by a method analogous to that by which we obtained the *sum* of an arithmetical series. For it follows from what is said in the text, that *the product of all the terms of a geometric progression, consisting of* n *terms, equals the square root of the* n*th power of the product of the two extremes.* But such a proposition is never needed in practice.

Let the series whose sum is required be—

2, 6, 18, 54, 162, 486, 1458, and let S = their sum.

Here the common ratio is 3. Now if we multiply every term of the series by the common ratio, we find that—

I. $3\,S = \quad 6 + 18 + 54 + 162 + 486 + 1458 + 4374.$

II. But $S = 2 + 6 + 18 + 54 + 162 + 486 + 1458.$

Here the same set of numbers occurs in both series, except the last term of (I.) and the first of (II.) Therefore subtracting (II.) from (I.) we have $2\,S = 4374 - 2 = 4372$; and $S = 2187$.

Or let the series be a, ar, ar^2, ar^3, ar^4, ... ar^{n-1}.

Then $S = a + ar + ar^2 + ar^3 + ar^4 + \ldots . ar^{n-1}$. (1)

And $Sr = \quad ar + ar^2 + ar^3 + ar^4 + ar^5 \ldots . ar^n$. (2)

Subtract (1) from (2), then $S\,(r - 1) = ar^n - a$.

General Formula.— $S = \dfrac{ar^n - a}{r - 1} = \dfrac{(r^n - 1)\,a}{r - 1.}$

In the case in which the common ratio was 3, and we multiplied the last term by 3 and subtracted the first term, the result gave *twice* the sum. In like manner had the common ratio been 6, six times the last term, minus the first term, would have given *five* times the sum. In (266) this principle was applied to the summation of recurring decimals, and then, as 10 or some power of 10 was always the common ratio, the result of the same operation always gave 9 or 99 or 999, &c., times the sum.

524. TO FIND THE SUM OF A GEOMETRIC PROGRESSION—

Find the last term by (517). Multiply this by the common ratio and subtract the first term. This answer divided by a number *one less* than the common ratio will give the sum.

Example I.—Find the sum of 7 terms of the series 2, 4, 8, &c.

Here because nth term $= ar^{n-1}$, and $r = 2$, and $n = 7$—

$\therefore$ (516) 7th term $= 2 \times 2^6 = 128$.

And because $S = \dfrac{(n\text{th term} \times r) - a}{r - 1}$

$\therefore$ Sum of 7 terms $= (128 \times 2) - 2 = 254$.

Example II.—Find the sum of 5 terms of the series 1, $\frac{2}{3}$, $\frac{4}{9}$, &c.

Here the common ratio is $\frac{2}{3}$.

$$\therefore \text{ the 5th term} = (\tfrac{2}{3})^4 = \tfrac{16}{81}.$$

$$\therefore S = \frac{\frac{16}{81} \times \frac{2}{3} - 1}{\frac{2}{3} - 1} = \frac{-\frac{211}{243}}{-\frac{1}{3}} = 2\tfrac{49}{81}.$$

Example III.—Sum the series ·43434343, *ad infinitum.*

This is equivalent to $\frac{43}{100} + \frac{43}{100^2} + \frac{43}{100^3} + \frac{43}{100^4}$, &c., and is therefore a series of progression by equal ratios, the common ratio being $\frac{1}{100}$.

Multiply every term by 100.

Then $100\,S = 43 + \frac{43}{100} + \frac{43}{100^2} + \frac{43}{100^3} + \frac{43}{100^4}$, &c.

Subtract $S = \frac{43}{100} + \frac{43}{100^2} + \frac{43}{100^3} + \frac{43}{100^4}$, &c.

Then $99\,S = 43$. Therefore $S = \dfrac{43}{99}$.

Example IV.—Sum the series $\frac{1}{4} + \frac{1}{16} + \frac{1}{64}$, &c., *ad infinitum.*

Here $\frac{1}{4}$ is the common ratio, we therefore multiply every term by 4.

$\therefore \quad 4\,S = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} +$ &c., *ad inf.*

Subtract $S = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} +$ &c., *ad inf.*

Subtract $3\,S = 1$ and $S = \frac{1}{3}$.

Exercise CXLIII.

Sum the following series :—

1. 1, 4, 16, &c., to 7 terms, and 3, $4\frac{1}{2}$, $6\frac{3}{4}$, to 5 terms.
2. 4, 3, $\frac{9}{4}$, to 10 terms, and $\frac{3}{2}$, 1, $\frac{2}{3}$, to 12 terms.
3. 5, 20, 80, to 8 terms, and 100, 40, 16, to 10 terms.
4. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, &c., *ad inf.*, and $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, *ad inf.*
5. Suppose a ball to be put in motion by a force which drives it 12 miles the first hour, 10 miles the second, and so on, continually decreasing in the proportion of 12 to 10 to infinity; what space would it move through?
6. If one grain of wheat be placed on the first square of a chess-board, two on the second, four on the third, and so on, doubling the number for each of the 64 squares, how many bushels of wheat would be required, supposing that 7,680 grains fill a pint measure?

525. We may trace a certain correspondence between the analogous expressions in the two kinds of progression. Thus—

I. Arithmetical Progression.	*II. Geometrical Progression.*
In which a=first term, d=common difference, and n=number of terms.	In which a= first term, r =common ratio, and n = number of terms.
(503) nth term $= a + \overline{n-1}\, d$.	(516) nth term $= a \times r^{n-1}$.
And $d = \dfrac{n\text{th term} - a}{n-1}$.	(520) $r = \sqrt[n-1]{\dfrac{n\text{th term}}{a}}$.
By (510) sum of all the terms $= \dfrac{(a + n\text{th term})\, n}{2}$.	By (522) product of all the terms $= \sqrt{(a \times n\text{th term})^n}$.

On comparing these two we see that—

Addition in (I.) corresponds to Multiplication in (II.)

Subtraction in (I.) corresponds to Division in (II.)

Multiplication by a given number in (I.) corresponds to Involution to a given power in (II.)

Division by a given number in (I.) corresponds to Evolution of a given root in (II.)

SECTION III.—LOGARITHMS.

526. The resemblances noticed in the last paragraph are of great importance in Arithmetic. They probably suggested to the mind of the inventor* the method of calculating by Logarithms, or of establishing such a connexion between numbers in arithmetical and others in geometrical progression, as shall enable us to deal with the latter by means of the simpler operations of the former, and substitute addition and subtraction in long and involved computations for multiplication and division. The one arithmetical truth on which the use of this method is founded, and which must be borne in mind throughout this chapter, has been already stated in (137).

527. *We multiply different powers of any number together when we add the exponents of those powers.*

* Baron Napier, or Neper, of Murchiston, in Scotland, published a book describing his discovery in the year 1614. The book was entitled "Mirifici Logarithmorum *Canonis Descriptio*"—An account of the marvellous System of Logarithms.

Demonstrative Example.—

$$5^3 \times 5^6 = (5 \times 5 \times 5) \times (5 \times 5 \times 5 \times 5 \times 5 \times 5) = 5^9.$$

$$a^2 \times a^5 = aa \times aaaaa = a^{2+5} = a^7.$$

We divide one power of a number by another power of the same number when we subtract the exponent of the divisor from that of the dividend.

Demonstrative Example.—

$$20^7 \div 20^3 = \frac{20 \times 20 \times 20 \times 20 \times 20 \times 20 \times 20}{20 \times 20 \times 20} =$$

$$20 \times 20 \times 20 \times 20 = 20^4 \text{ or } 20^{7-3}.$$

$$a^9 \div a^6 = \frac{aaaaaaaaa}{aaaaaa} = aaa = a^3 = a^{9-6}.$$

528. Let us now form the simplest series in arithmetical progression, beginning with 0, and having 1 for the common difference;

And also a simple geometric progression, beginning with 1, and having 3 for the common ratio.

I.	0	1	2	3	4	5	6	7	8	9	10	11.
II.	1	3	9	27	81	243	729	2187	6561	19683	59049	177147.

Here the numbers in the upper line are called the *logarithms* of those which are placed beneath them.

Observation.—*Logarithm*, from λόγων ἀριθμός, = the number of the ratios. Thus 7 is the logarithm of 2187, because it represents the *number of the ratios* or powers of three, which are contained in 2187, and 10 is the logarithm of 59049.

529. The exponent in any numerical expression may therefore be considered as a logarithm.

Thus $a^x = b$. Here x is the logarithm of b* to the base a, or it expresses the *number of the ratios* or powers of a which are contained in b.

530. Suppose it be required to multiply two numbers in the lower series together, 243 and 729 for instance; we notice that above them are the numbers 5 and 6. Now $5 + 6 = 11$, and under 11 we may find 177147, or the product of 243 and 729.

This would evidently be the case however far the series might be extended, for all the numbers in the lower line represent powers of 3,

* It is usual to contract the expression, thus: log. b means logarithm *of* b.

and the figures above them are the exponents of those powers.

And because $243 = 3^5$, and $729 = 3^6$,

$$\therefore 243 \times 729, \text{ or } 3^5 \times 3^6 = 3^{11} = 177147.$$

531. In like manner we may divide any one of the terms in the lower series by any other which is less than itself, if we subtract the exponent of the divisor from that of the dividend, and take the number which stands underneath the difference.

Thus $59049 \div 729 = 81$.

Because $59049 = 3^{10}$, and $729 = 3^6$,

$$\therefore \text{by } (527)\ \frac{3^{10}}{3^6} = 3^4 = 81.$$

532. We may construct any other series fulfilling the conditions described in 528, and obtain similar results. *e.g.*,

0	3	6	9	12	15	18	21	24	27	30	33	36	39.
1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192.

Take any two numbers from the lower series—32 and 128. Now find a number as many places to the right of 128 as 1 is to the left of 32.

Then $1 : 32 :: 128 : 4096$.

Now by the law of proportion (319)

$$1 \times 4096 = 32 \times 128.$$

But above these numbers are 0, 15, 21, 36, of which the second exceeds the first as much as the fourth exceeds the third.

And by the law of arithmetical progression (55)

$$0 + 36 = 15 + 21.$$

But $36 = \log. 4096$, and $15 = \log. 32$, and $21 = \log. 128$.

Hence $\log. (32 \times 128) = \log. 32 + \log. 128$.

533. The following are obvious inferences from this reasoning:—

I. The sum of the logarithms of two or more numbers equals the logarithm of their product.

General Formula.—$\text{Log. } abc = \log. a + \log. b + \log. c$.

And $\log. a^n = n \times \log. a$.

II. The difference between the logarithms of two numbers equals the logarithm of their quotient.

General Formula.—$\text{Log. } \frac{a}{b} = \log. a - \log. b$, and $\log. \sqrt[n]{a} = \frac{\log. a}{n}$.

534. These relations once established between any two series of numbers, it follows that as far as those series extend, we may be spared the trouble of multiplying and dividing numbers whose logarithms are known, and may add or subtract those logarithms instead.

It thus becomes desirable to construct a table which shall comprise all ordinary numbers, and shall give the logarithms of each. For this purpose, any number may be chosen as the base. *e.g.*,

I.	0	1	2	3	4	5	6	7	8	9.
	1	2	4	8	16	32	64	128	256	512.

II.	0	1	2	3	4	5	6	7.
	1	5	25	125	625	3125	15625	48625.

Of these systems, (I.) is constructed with the base 2, and (II.) is constructed with the base 5.

535. *Observation.*—No use can be made in calculation of such tables as these, however far they may be carried; because the numbers whose logarithms are given are very few, and the tables contain no provision for dealing with any numbers unless they happen to be expressed with integral exponents as powers of the base. It is necessary, therefore, whatever system be adopted, to calculate the fractional exponents of all the intermediate numbers.

536. Two or three important facts require to be noticed here.

I. The logarithm is, in every case, the differential mean between the two logarithms, one on each side of itself.

II. The number in the lower line is, in every case, the proportional mean between the two numbers, one on each side of itself.

Thus, in series (II.), 3, 4, and 5 are in equi-different progression, while 125, 625, 3125, are in equi-rational progression.

$\therefore$ by (508) $\frac{3 \times 5}{2} = 4 =$ the differential mean between 3 and 5.

And by (519) $\sqrt{3125 \times 125} = 625 =$ the proportional mean between 125 and 3125.

537. In like manner, if any two terms be taken at random in the series of logarithms, the other logarithms between them form a series of differential means; while the numbers to which the logarithms belong form a series of proportional means between the two extremes

Thus, in (I.), we take from the upper line 3 and 8.

Now the logarithms between these, viz., 4, 5, 6, and 7, are 4 differential means between 3 and 8 (505).

But the numbers of which 3 and 8 are logarithms, are 8 and 256.
And the series of numbers between them, are 16, 32, 64, and 128.
And these are four proportional means between 8 and 256.
Hence we infer that,

538. *If we take any two numbers whose logarithms are known, and find a mean proportional between them, the result will be a number whose logarithm will be a differential mean between the two logarithms.*

For the differential mean between 2 and 3 is $2\frac{1}{2}$, or 2·5.

Hence in (I.) 2·5 is the log. of $\sqrt{4 \times 8}$, or the proportional mean of those numbers whose logs. are 2 and 3.

Also in (II.) 2·5 is the log. of $\sqrt{25 \times 125}$ for the same reason.

539. The system of logarithms now universally adopted is the decimal.* It was invented by Briggs, and is commonly called his system; it is as follows:—

0	1	2	3	4	5	6	7	8,	&c.
1	10	100	1000	10000	100000	1000000	10000000	100000000,	&c.

In order to construct the common tables on this basis, it becomes necessary *to fill up the lower of these lines with the set of natural numbers; to consider each of them as a term in geometrical proportion; and to connect each with another number, which should show what term it is in the series.* The process of discovering the logarithms of the intermediate numbers is a very difficult and laborious one, involving the use of some of the most advanced truths in mathematical science. The logarithms of any numbers which are not powers of 10 are surds, and can only be approximately expressed.

* The inventor of logarithms was led to his discovery by the desire to simplify geometrical and astronomical formulæ. He did not fully anticipate their extensive application to ordinary arithmetical processes. Hence the *radix* or *base* of his scale was determined for certain geometrical reasons. We cannot enter into them in this place, but it is worth remembering, that in the Napierian or Hyperbolic system, the logarithm of 10 is 2·3025852; the number whose logarithm is 1 being 2·7182818. This latter number is called the *base* of the Hyperbolic or Napierian Logarithms; and it is evident that any logarithm on the decimal system, multiplied by the former number, will give the logarithm on the Hyperbolic. Henry Briggs published, in the year 1617, the book explaining the change which he proposed to make in the base.

540. Suppose now that by the higher processes we have referred to, it be found that the logarithm of 5 may be approximately expressed thus—

$$\text{Log. } 5 = \cdot 69897.$$

This means that $10^{\cdot 69897} = 5$, or that ·69897 is the exponent which represents what power of 10 the number 5 is.

We may obtain many inferences from this fact. *e.g.*,

I. From (533) it follows that—

Log. $5 \times 5 =$ log. $5 +$ log. $5 = 2$ log. 5.

Or log. $25 = 2 \times \cdot 69897 = 1\cdot 39794$.

II. And log. $10 \div 5 =$ log. $10 -$ log. 5.

But log. $10 = 1$, $\quad \therefore$ log. $2 = 1 - \cdot 69897 = \cdot 30103$.

III. Because $8 = 2 \times 2 \times 2$, $\therefore$ log. $8 = 3$ log. 2.

$\therefore$ from (II.) log. $8 = 3 \times \cdot 30103 = \cdot 90309$.

IV. And log. $16 =$ log. $2 +$ log. $8 = \cdot 30103 + 90309 = 1\cdot 20412$.

V. Log. $\sqrt{16}$ or log. $4 = \dfrac{\text{log. } 16}{2} = \dfrac{1\cdot 20412}{2} = \cdot 60206$.

VI. And log. $5^7 = 7$ log. $5 = \cdot 69897 \times 7 = 4\cdot 87479$.

541. Because the logarithms of the several powers of 10 are integer numbers, it follows that if the logarithm of a number be known, the logarithm of that number, multiplied or divided by any power of 10, will differ from it in the whole number, not in the fractional part. The whole number is called the *characteristic*, and the fractional part the *mantissa*.

Thus because—			log. 25 =	1·39794
Log. 250	= log. 10	+	log. 25 =	2·39794
Log. 2500	= log. 100	+	log. 25 =	3·39794
Log. 25000	= log. 1000	+	log. 25 =	4·39794

Hence also log. 25 ÷ 10,	or log.	2·5 =	·39794
Log. 25 ÷ 100,	or log.	·25 =	·39794 − 1, or $\bar{1}\cdot 39794$
Log. 25 ÷ 1000,	or log.	·025 =	·39794 − 2, or $\bar{2}\cdot 39794$
Log. 25 ÷ 10000,	or log.	·0025 =	·39794 − 3, or 3·39794
Log. 25 ÷ 100000,	or log.	·00025 =	·39794 − 4, or $\bar{4}\cdot 39794$

Observation.—On referring to a book of tables to find the logarithm either of 25, or 25000, or ·025, or any of the numbers just given, the only figures found would be 39794. The characteristic is never given, and is determined by the following considerations:—

542. I. From (539) it will be seen that the characteristic of every number between 1 and 10 is 0, that of every number between 10 and 100 is 1, that of every number between 100 and 1000 is 2, and generally *the characteristic of every integer number is the same as the number of its digits, minus one.*

543. II. The *characteristic* of a decimal fraction or of a number divided by 10, or some power of 10, must always be subtracted, and is written with the sign of subtraction over it, thus: "log. ·0025 = $\bar{3}$·39794" means that though the mantissa may be added, the characteristic must be subtracted from any other logarithm which may require to be connected with it. If the first significant figure of the number be in the first decimal place, the characteristic is $\bar{1}$; if it be in the second the characteristic is $\bar{2}$; and generally, *the characteristic of any decimal fraction has the minus sign above it, and is the number which represents the place of the first significant figure to the right of the decimal point.*

544. Since (533) the logarithm of any composite number can readily be deduced from the logarithms of its factors, if the logarithms of prime numbers be once ascertained, those of all other numbers can readily be found. The following is a table of the logarithms of all the prime numbers up to 100, expressed as in ordinary books without the characteristic:—

Prime Numbers.	Logarithms.	Prime Numbers	Logarithms.
2	30103	43	6334685
3	4771213	47	6720979
5	69897	53	7242759
7	845098	59	770852
11	0413927	61	7853298
13	1139434	67	8260748
17	2304489	71	8512583
19	2787536	73	8633229
23	3617278	79	8976271
29	462398	83	9190781
31	4913617	89	94939
37	5682017	97	9867717
41	6127839		

545. To find the logarithm of a number—

RULE.

Resolve the number into its prime factors and add the logarithms of those factors together.

Or, if the number be a fraction, subtract the logarithm of the divisor from that of the dividend, and the remainder is the logarithm of the quotient or of the fraction.

Or, if the number be a power of a known number, multiply the logarithm of that known number by the exponent of the power.

Or, if the number be a root of a known number, divide the logarithm of that known number by the index or exponent of the root.

Example.—From the logs. of 2 and 7 deduce the logs. of 2800, of 3·43, $12\frac{1}{4}$, and of $\sqrt[3]{14}$.

1. Because $2800 = 2 \times 2 \times 7 \times 100$,
$\therefore$ log. $2800 =$ log. 2 + log. 2 + log. 7 + log. 100.
$\therefore \qquad = \cdot 30103 + \cdot 30103 + \cdot 845098 + 2 = 3\cdot 447158.$

2. Because $3\cdot 43 = 7 \times 7 \times 7 \times \frac{1}{100}$,
$\therefore$ log. $3\cdot 43 = 3$ log. 7 $-$ log. 100.
$\therefore \qquad = 3 \times \cdot 845098 - 2 = \cdot 5352941.$

3. Because $12\frac{1}{4} = \frac{49}{4}$, $\therefore$ log. $12\frac{1}{4} =$ log. 49 $-$ log. 4.
But log. $49 = 2$ log. $7 = 1\cdot 690196$, and log. $4 = 2$ log. $2 = 60206$.
$\therefore$ log. $12\cdot 25 = 1\cdot 690196 - 60206 = 1\cdot 088136$.

4. Because $\sqrt[3]{14} = \sqrt[3]{2 \times 7}$, $\therefore$ log. $\sqrt[3]{14} = \dfrac{\text{log. } 2 + \text{log. } 7}{3}$.
$= (\cdot 30103 + \cdot 845098) \div 3 = \cdot 3820426.$

EXERCISE CXLIV.

1. From logs. 2 and 3 deduce the logs. of the numbers 6, 27, 3·6, 1600, 6·4, 2·25, $\sqrt{2}$, $\sqrt[3]{12}$, $\cdot\dot{3}$, and $\cdot\dot{8}$.

2. From logs. 5 and 7 deduce the logs. of ·35, of 49000, of 6·25, of 17·5, of $\frac{14}{25}$, of $1\frac{3}{7}$, of 2450000.

3. From logs. of 7 and of 2, deduce those of 1400, of 5·6, of ·00196, and of 980.

4. Given logs. 3 and 2, deduce from them those of 60, of 27000, of 1620, of 405, of 3·6, and of 4800.

5. Given logs. 5, 11, and 3, deduce logs. 605, 1665, ·0045, of $\frac{11}{15}$, of 726, $91\cdot\dot{6}$, 44, 6·75, 198, 24·2, ·10264.

546. *Calculation of Differences.**—Ordinary tables only give the logarithms of numbers consisting of either 4 or 5 digits, the logarithms themselves being generally carried to the 6th or 7th place. If we desire to find the logarithm of 254329, and on looking into a table find that the logarithms of 2543 and of 2544 are given, but not of any intermediate numbers, the number must be calculated in the following manner:—

Log. 2543 = ·40535, and log. 2544 = 40552.

The difference between these logarithms is 17.

Now the difference between 254300 and 254400 is 100, and the difference between 254300 and 254329 is 29. Hence we have the following proportion:—*If for* 100, *difference between* 254300 *and* 254400, *we have* 17 *difference between their logarithms, what should be the difference between the logarithms of the two numbers* 254300 *and* 254329, *which differ by* 29?

The proportion is $100 : 29 :: 17 : x$. And $x = \frac{29 \times 17}{100} = 4\cdot93$.

Hence log. 254329 = 5·40535 + 4·93 = 5·40540.

547. *Observation.*—The difference is always given in a separate column in the tables, and is called the tabular difference. The proportion thus established between the differences of numbers and the differences of their logarithms is not strictly true, but gives a result sufficiently near for all practical purposes.

548. When the logarithm of the answer to a sum is found it becomes necessary to find the number of which it is the logarithm. To do this we must look in the table for the number, omitting the characteristic. For example: if the answer to a sum were to be 2·3991717 we should look in the table for ·3991717, and having

* Throughout the remainder of this section it is assumed that the student has a book of tables at hand. Hutton's and Babbage's are the completest; but any set will answer the purpose.

found the number 25071 as the natural number belonging to it, we should then use the characteristic 2, in order to determine the value of these figures. Now, since 2·3991717 must by (541) be the logarithm of a number between 100 and 1000, the decimal point must be placed after the third figure, and 2·3991717 = log. 250·71.

In this case the given logarithm was found exactly in the tables; but this does not always happen, and it often becomes necessary, therefore, to calculate the difference. Thus: suppose it is required to find the number corresponding to 6·45936; we look in a table and find that log. 2879 = 45924, and log. 288 = 45939, the tabular difference being 15: we may then make this proportion—*If for* 15 *difference between the logarithms of* 2879 *and* 2880 *there be a difference of one in the numbers themselves, what should be that number whose logarithm differs from that of* 2879 *by* 11?

$$15 : 11 :: 1 : x, \quad \text{and } x = \tfrac{11}{15} = \cdot 8.$$

Hence ·45936 = log. 28798, and 6·45936 = log. 2879800.

The following examples, worked by the help of logarithmic tables, will show how they are applicable to involved problems in arithmetic:—

549. *Example I.*—Simplify the expression $\dfrac{472{\cdot}8^3 \times \sqrt[5]{268450}}{1876 \times 327^2}$.

Now by (545) the logarithm of the answer to this sum will be—

$$3 \text{ log. } 472{\cdot}8 + \frac{\text{log. } 268450}{5} - (\text{log. } 1876 + 2 \text{ log. } 327).$$

But log. 472·8 = 2·6746775. Hence log. $472{\cdot}8^3$ = 8·0240325
And log. 268430 = 5·4288634. Hence log. $\sqrt[5]{268450}$ = 1·08577268
Logarithm of numerator = 9·10980518

Log. 1876 = 3·2732328
2 log. 327 = 2 × 2·5145478 = 5·0290956
Logarithm of denominator = 8·3023284 = 8·3023284
Log. 64191 = ·80747678

But since the characteristic of the logarithm is 0 the answer lies between 1 and 10, and is therefore 6·4191.

By logarithms we may readily solve those questions in Geometrical Progression (521) for which the ordinary processes of Evolution will not suffice.

550. *Example II.*—Suppose it be required to insert 6 proportional means between 2 and 4374. Now here it is first necessary to find the common ratio of the series, which consists of 8 terms.

By (524) $r = \sqrt[n-1]{\frac{n\text{th term}}{a}}$ $\therefore r = \sqrt[7]{\frac{4374}{2}} = \sqrt[7]{2187}$.

$\therefore$ log. r = log. $2187 \div 7 = \frac{3\cdot3398488}{7} = \cdot477121$ = log. 3.

Hence 3 is the common ratio, and 6, 18, 54, 162, 486, 1458 form the series required.

Example III.—Insert 25 proportional means between 3 and 4.

Here $r = \sqrt[26]{\frac{4}{3}}$, and log. $r = \frac{\text{log. } 4 - \text{log. } 3}{26}$.

But log. 4 − log. 3 = ·60206 − 4771213 = ·1249387.

Dividing this by 26, log. r = ·00480 = log. 1·0111.

The first term, 3, multiplied by this number will give the 2nd term, and multiplied by its square will give the 3rd term. To find any term, say the 11th term of the series—

By (517) 11th term $= ar^{10}$, $\therefore$ log. 11th term = log. 3 + 10 log. 1·0111 = 52517 = log. 3·351. This number is the 10th mean proportional, or the 11th term of the series.

551. It is occasionally required to solve questions by the aid of logarithms which take this form :—

I. To what power must 5 be raised that it may equal 20?

Here if x = the unknown exponent, $5^x = 20$.

$\therefore x$ log. 5 = log 20, and $x = \frac{\text{log. } 20}{\text{log. } 5} = \frac{1\cdot30103}{\cdot69897} = 1\cdot861$.

Hence $5^{1\cdot861} = 20$, or 1·861 is the exponent which raises 5 to 20.

II. What would be the logarithm of 180 if the base were 12?

Here if x be the required logarithm, $12^x = 180$.

$\therefore x$ log. 12 = log. 180, $\therefore x = \frac{\text{log. } 180}{\text{log. } 12} = \frac{2\cdot2552726}{1\cdot0791813} = 2\cdot089$.

And $12^{2\cdot089} = 180$, or 2·089 would be the logarithm of 180 on the base 12.

Application of Logarithms to Interest and Annuities.

552. *Compound Interest.*—It was stated in (381) that compound interest was to be found by working a separate sum for each of the periods mentioned in the question. The rule of Progression, however enables us to abridge this laborious operation. For if the rate of interest on a sum of money be uniform through a series of years, the amounts at the end of each period will form a series in geometrical progression, and as the amount at the end of the first year is to the amount at the end of the second year, so is that amount to the amount at the end of the third year, &c. And if A, A_2, A_3, &c., represent the several amounts, it will be true that—

$A_1 : A_2 : A_3 : A_4 : \ldots\ldots A_n$ are in continued proportion.

We have first to find the constant ratio of this series.

Now if £1 be put out at interest at R per cent., at the end of the first year it will amount to $1 + \frac{R}{100}$ or $\frac{100 + R}{100}$; let us call this amount A.

Then at the end of the second year the sum will amount to $A + \frac{AR}{100}$ or $\frac{100\,A + AR}{100}$ or $A \times \frac{100 + R}{100}$.

But $A = \frac{100 + R}{100}$, $\therefore$ amount at the end of second year $=$ $\frac{100 + R}{100} \times \frac{100 + R}{100} = \left(\frac{100 + R}{100}\right)^2$; call this amount A''.

Then at the end of the third year the amount will be $A'' + \frac{A''R}{100}$ $= \frac{1000\,A'' + A''R}{100} = A'' \times \frac{100 + R}{100}$.

But $A'' = \left(\frac{100 + R}{100}\right)^2$, $\therefore$ amount at end of third year $=$ $\left(\frac{100 + R}{100}\right)^3$ or $\left(1 + \frac{R}{100}\right)^3$.

And since it is evident that $\frac{100 + R}{100}$ or $1 + \frac{R}{100}$ is the constant ratio of this series, it follows that—

Amount of £1 for n years at R per cent. $= \left(1 + \frac{R}{100}\right)^n$.

Amount of £P for n years at R per cent. $= P\left(1 + \frac{R}{100}\right)^n$.

RULE FOR COMPOUND INTEREST.

553. Find in a decimal form the sum to which £1 would amount at a given rate in one year.* Involve this number to the power representing the number of years, and multiply by the principal.

554. *Example I.*—Find the compound interest on £12000 at 4 per cent. for 7 years.

By the formula, Amount $= 12000\left(1 + \frac{4}{100}\right)^7 = 12000 \times 1{\cdot}04^7$.

$$\therefore \text{log. A} = \text{log. } 12000 + 7 \text{ log. } 1{\cdot}04.$$

$$\begin{aligned} \text{But } 7 \text{ log. } 1{\cdot}04 = 7\ ({\cdot}0170333) &= {\cdot}1192331 \\ \text{log. } 12000 &= 4{\cdot}0791812 \\ \text{log. } 15791{\cdot}3 &= 4{\cdot}1984143 \end{aligned}$$

Answer—£15791 6s. = Amount, and £3791 6s. = Interest.

555. *Example II.*—Find the amount of £7 at compound interest at 3 per cent. for 100 years.

Here Amount $= 7 \times 1{\cdot}03^{100}$.

Log. A = log. 7 + 100 log. 1·03.

$$\begin{aligned} \text{But log. } 7 &= {\cdot}845098 \\ 100 \text{ log. } 1{\cdot}03 &= 1{\cdot}28372 \\ \text{Log. } 134{\cdot}53 &= 2{\cdot}128818 \end{aligned}$$

Hence £134 10s. 7½d. = Amount.

556. *Example III.*—In what time will a sum of money double itself at 5 per cent?

Here, because Amount $= \left(1 + \frac{R}{100}\right)^n$

$$\therefore \text{log. A} = n \times \text{log. }\left(1 + \frac{R}{100}\right)$$

$$\text{And } n = \frac{\text{log. A}}{\text{log. }\left(1 + \frac{R}{100}\right)}$$

But if £1 be taken, $1 + \frac{R}{100} = 1{\cdot}05$, and $A = 2$.

$$\text{And } \frac{\text{log. } 2}{\text{log. } 1{\cdot}05} = \frac{{\cdot}3010300}{{\cdot}0211893} = 14{\cdot}20606 \text{ years.}$$

* If the payments be made half-yearly or otherwise the amount should be calculated for that period, and n will represent the number of half-years or intervals of payment.

ANNUITIES.

557. An Annuity is a sum of money paid at yearly intervals, and may arise from estates, from invested capital, from a pension, or any other source. Thus: the lease of an estate worth £80 a year, and which will expire in 35 years, is to the owner an annuity of £80 for 35 years.

Annuities which are to last for a fixed term of years are called *certain*, and those which are only to last during the lifetime of any particular person or persons are called *contingent*. If an annuity becomes payable at once, it is said to be *immediate*; if it is only to be entered upon at a certain distant period, it is called a *reversionary* or *deferred* annuity.

The *amount* of an annuity means the sum of all the payments, together with the interest upon them, from the time at which they become payable until the expiration of the whole term.

Now if an annuity be regularly paid at the end of each year for 20 years, interest upon the first payment accrues for 19 years; the second payment is subject to interest for 18 years, and it is only the last payment, viz. that paid at the end of the 20th year, on which no interest has to be calculated.

Now suppose interest be at 5 per cent., and that the annuity be of £1; then the last payment = £1 only; the payment of the year before is £1 + its interest for a year; that of the eleventh year requires nine years' compound interest to be added, and the several sums of money with their respective interests form a series.

20th year	19th	18th	17th	16th		1st
£1,	1·05,	$1{\cdot}05^2$,	$1{\cdot}05^3$,	$1{\cdot}05^4$,	......	$1{\cdot}05^{19}$

This series shows us each year's principal with its compound interest for the time which has to expire before the termination of the 20 years. Now the series is one in geometrical progression, the common ratio being 1·05.

Hence the total amount of the annuity is the sum of this series.

And by (524) Sum $= \dfrac{ar^n - a}{r - 1} = \dfrac{1{\cdot}05^{20} - 1}{1{\cdot}05 \;\; - 1}$.

$$1{\cdot}05^{20} - 1 = 2{\cdot}65329771 - 1 = 1{\cdot}65329771.$$

$$1{\cdot}65329771 \div 1{\cdot}05 - 1 \qquad £33{\cdot}065954 = £33 \text{ 1s. } 3\tfrac{3}{4}\text{d.}$$

It follows, that if the annuity had been of £25, or of £60, or any number of pounds per annum; this sum, £33 1s. 3¾d., which represents the amount of an annuity of £1, would require to be multiplied by £25, or by £60, and that thus the amount of any such series of annual payments may always be found.

558. To FIND THE AMOUNT OF AN ANNUITY—

Find by (553) the amount of £1 at compound interest for the given time, subtract unity from this amount, multiply this difference by the annuity, and divide this product by one year's interest on £1.

Example.—What will an annuity of £48 5s. amount to in 15 years at 4 per cent.?

Here if a be the annuity the answer must be $a\left(\frac{r^n - 1}{r - 1}\right)$

But $a = £48\ 5s.$, $n = 15$, and $r = 1{\cdot}04$, $r - 1 = {\cdot}04$.

And the problem to solve is $\left(\frac{1{\cdot}04^{15} - 1}{{\cdot}04}\right) \times 48{\cdot}25$.

$$
\begin{array}{rr}
\text{Now by logarithms* } 1{\cdot}04^{15} = & 1{\cdot}80094351 \\
\text{Subtract} & 1 \\
\hline
& {\cdot}80094351 \\
& 48{\cdot}25 \\
\hline
& 400471755 \\
& 160188702 \\
& 640754808 \\
& 320377404 \\
\hline
{\cdot}04) & 38{\cdot}6455243575 \\
\hline
\text{Answer—£966 2s. } 9\tfrac{1}{4}\text{d.} & 966{\cdot}13810898
\end{array}
$$

559. Sometimes the amount of the annuity is given, and the time, and it is required to find what the annuity itself is. In this case it is only necessary to find what would be the amount of an annuity of £1, for the given time, and divide the whole given sum by that amount.

Example.—Suppose money can be improved at 3 per cent. compound interest, how much ought I to lay by per annum in order to be worth £1000 eleven years hence?

* These results are, in practice, more frequently obtained from Annuity Tables which have been prepared especially for this purpose, and which save the trouble of working each part of the sum by logarithms.

Here by the former rule, a sum of £1 laid by per annum at 3 per cent. for 11 years, will amount to 12·807796.

Hence the required sum is $\frac{1000}{12{\cdot}8078} = 78{\cdot}077 = £78\ 1s.\ 6\tfrac{1}{2}d.$

Exercise CXLV.

1. Find the amount of an annuity of £325 in 9 years at £3 6s. per cent. compound interest.

2. Find the amount of an annuity of £30 6s. 4d. for 12 years at 5 per cent. compound interest.

3. Find the amount of an annuity of £24 at 3½ per cent. for 29 years.

4. Find the amount of an annuity of £283 at 6 per cent. for 28 years.

5. Find the amount of an annuity of £72 at 5½ per cent. for 17 years 6 months.

6. What sum must be put by annually for 30 years, in order to amount to £500, if money is worth 3¾ per cent.?

560. The PRESENT VALUE of an annuity is determined by the same considerations as apply to the deduction of discount (384). If £1 be payable a year hence, its present value is such a sum as, if improved at the current rate of interest, would amount to £1 at the end of the year. Hence—

Present value : £1 :: £1 : £1 + interest.

Suppose the rate of interest be 6 per cent.; then £1·06 due a year hence is worth £1 at this moment, and

$\therefore$ £1 due a year hence is now worth $\frac{1}{1{\cdot}06}$.

And because (552) £1 will be worth £$1{\cdot}06^2$ two years hence, at compound interest,

$\therefore$ present value of £1 due two years hence $= \frac{1}{1{\cdot}06^2}$.

In like manner it may be shown that—

Present value of £1 due three years hence $= \frac{1}{1{\cdot}06^3}$.

Present value of £1 due four years hence $= \frac{1}{1{\cdot}06^4}$.

Hence if a be the annuity, its present value for a number of years is represented by the geometrical series—

$$\frac{a}{1{\cdot}06};\quad \frac{a}{1{\cdot}06^2};\quad \frac{a}{1{\cdot}06^3};\quad \frac{a}{1{\cdot}06^4};\quad \frac{a}{1{\cdot}06^5};\quad \frac{a}{1{\cdot}06^6};\ \ldots\ldots \text{\&c.},$$

of which the common ratio is $\frac{1}{1{\cdot}06}$, and the Sum of the series, as determined by the rule in (524), is equal to—

$$\left(\frac{1-\frac{1}{1{\cdot}06^n}}{{\cdot}06.}\right)\times a.$$

561. *Observation.*—The number representing the present value of £1, due any number of years hence, is the reciprocal of the number representing the sum to which it would amount if invested at compound interest from the present time to the expiration of the time. For compound interest on £1 at 6 per cent. for 20 years $= 1{\cdot}06^{20}$. And present value of £1 at 6 per cent. 20 years hence $= \frac{1}{1{\cdot}06^{20}}$.

562. To find the present value of an annuity—

Divide £1 by the sum to which it would amount at compound interest for the given time. This will give the present value of £1. Subtract this sum from unity, multiply this difference by the annuity, and divide the result by the interest on £1 for one year.

Example.—Find the present value of an annuity certain, of £50, for 18 years, at 3½ per cent.

Present value of £1 due 18 years hence $= \frac{1}{1{\cdot}035^{18}} = {\cdot}53836114$.

$$\begin{array}{lcr} 1 - {\cdot}53836114 & = & {\cdot}46163786 \\ & & 50 \\ & {\cdot}035) & \overline{23{\cdot}081893} \\ & & \overline{659{\cdot}4826} \end{array}$$

Answer—£659 9s. 7¾d.

Exercise CXLVI.

1. If money be improvable at 5 per cent., what is the present value of an annuity of £170 for 20 years?

2. At $2\frac{1}{2}$ per cent. find the present value of an annuity of £100 for 46 years.

3. At 6 per cent. what is the present value of an annuity of £25 for 34 years?

4. At $4\frac{1}{2}$ per cent. what is the present value of an annuity of £234 for 16 years?

5. What sum ought to be expended in the purchase of an annuity of £150 for a person 43 years of age, supposing the average expectation of such a life extends to 58 years, and that money is worth only 3 per cent?

6. How much ought to be given for the lease of a house, if 27 years of the term be unexpired, and the yearly rental £74, calculating interest at 4 per cent.?

MISCELLANEOUS EXERCISES ON PROGRESSION AND LOGARITHMS.

Solve the following expressions:—

$$\sqrt[5]{\frac{16}{13}} - \sqrt[6]{\frac{5}{9}} \qquad \sqrt[6]{25} \times \sqrt[5]{347^4}$$

$$\sqrt{\frac{\sqrt[4]{32} \times \sqrt[3]{48}}{2\sqrt{27}}} \qquad \frac{1834^3 \times 796^7}{2584^5}$$

1. In what time will an annuity of £200 pay off a debt of £4000 at 3 per cent. compound interest?

2. Find x in the following expressions:—

$$8^x = 100$$
$$10^x = 2$$

3. In what time will a sum become ten times its original value at compound interest at 5 per cent.?

4. Insert 3 geometric means between 2 and $\frac{1}{8}$. What number is that which being raised to the 5th power will equal $\left(\frac{23}{417}\right)^3$?

5. In what time will a sum of money double itself at $3\frac{1}{2}$ per cent. compound interest?

6. In what time will £20 amount to £90 at 5 per cent.?

7. In how many years will £325 amount to £395·0394 at 5 per cent. compound interest?

8. A person invests £5000 in the 3 per cent. consols. when stocks are at 90, what will this sum amount to in 15 years, supposing the half-yearly interest, as it becomes due, to be invested at the same rate?

9. Solve the following expressions by the help of logarithmic tables :—

$$\sqrt[7]{\frac{\cdot 006 \times 625^{6}}{625}}; \qquad \sqrt[125]{\frac{125^{125}}{\cdot 125}}$$

10. To what power must 50 be raised to equal 1000?

11. To what power must $\frac{23}{417}$ be raised to equal ·17577?

12. Find a fourth proportional to the 19th power of 11, the 11th power of 19, and the 17th power of 17.

13. The mean distances of Mercury and Uranus are in the proportion of ·387098 to 19·1823900, and the time of the revolution of Mercury is 87·969258 days; what is the time of revolution of Uranus' the square of the times being as cubes of the mean distances?

14. In what time would £10 amount to £100 at 3 per cent. compound interest?

15. How much will be the discount on £1000 at 4 per cent. for two months?

16. If the discount on a promissory note of £500 1s. 3d. amounted to £32 1s. 3d., and the rate of interest $4\frac{1}{2}$ per cent. compound, how long had the note to run?

17. A debt of £550 accumulating at the rate of 3 per cent., is paid off by yearly instalments of £25; when is the debt discharged?

18. What should be paid for an annuity of £25 for 14 years, to commence at the end of 7 years, allowing $4\frac{1}{2}$ per cent. compound interest?

18. An annuity of £30 for 25 years is sold for £350; what interest is allowed?

19. A lease for 99 years is purchased for £100; what rent must be paid to you to allow $5\frac{1}{2}$ per cent.?

20. Multiply 79368 by 27415, and divide the product by 827·145.

21. Find a fourth proportional to $726^{3} : 298^{4} :: \sqrt{3072} : x$.

22. A lease for 500 years is purchased for £300, the purchaser not to receive anything from it for 20 years; what rent should then be obtained to make 6 per cent. interest?

APPENDIX.—A.

MONEY, WEIGHTS, AND MEASURES.

I. The Unit or Standard of Measurements.

It was shown in (4) that before Arithmetic can be applied to the measurement of any concrete magnitude, the *unit* of that magnitude must be clearly defined and understood. But such quantities as *length*, *solidity*, *weight*, *time*, and *value*, being capable of continuous increase, offer us no units which are clearly distinguishable, and by means of which they can be readily compared with one another. Men are compelled therefore to select arbitrary standards; and it is not surprising that almost every nation has its own peculiar method of measuring, and its own set of tables. On this account there is great difficulty in comparing ancient or foreign measures with our own.

Of all the magnitudes just mentioned, time is the easiest of measurement;* because the period of the earth's revolution round its axis, or one day, and the period of the earth's revolution round the sun, or one year, are portions of time which, as far as we know, never vary. They form the natural standards by which all nations alike measure their time. The day is the principal unit of time in all countries of the world, and all other periods are either aliquot parts, or multiples of a day. Nations may differ as to the sub-divisions of a day, but they cannot differ as to the day itself. Hence there is no difficulty in comparing foreign methods of computing time with our own; because nature has provided all men alike with a fixed unalterable unit to serve as the basis of their calculations.

It is to be desired that some natural standards could be found in like manner, to which lengths, values, bulks, weights, and other concrete magnitudes could be universally referred. But it is very difficult to find such standards in nature, and all the inconsistencies and difficulties of tables arise out of the fact that the standards in common use have been arbitrarily chosen, not on any system, but chiefly by accident. For instance, an object in nature which is always precisely of the same length, and which will therefore serve as the unit of linear measurement, is not easy to find. If it were found, it would be possible to form from it a unit of surface, and so to found square measure upon long measure. Then from the square

* It may be noted that, if it were not for the circumstance mentioned in the text, time would be of all magnitudes the hardest to measure. For in comparing magnitudes of any other kind, equality or inequality can be determined by bringing them together, and observing whether they coincide or not. This is manifestly impossible in the case of two periods of time; and but for the uniformity of the laws of nature we should find it impossible to verify our calculations respecting time.

unit of surface, we might obtain a cubic unit, or fixed standard of bulk, and this would serve also as a measure of capacity. Then, if some substance were chosen whose weight never varied, enough of this substance to fill a certain measure might be taken to serve as the unit of weight. In this way, superficial measure, measures of capacity (including ale, beer, wine, and dry measures), and measures of weight, could all be connected with the unit of length, and one fixed standard of length would suffice to determine all these measurements in a permanent way. We shall see that our own and most other systems rest to some extent on this principle, and that the fundamental difficulty is to determine the *linear* unit.

II. Measures of Value—Money.

Measures of value are of all others the least easy to compare and to understand. For there is nothing in nature which is always exactly of the same worth to men, or for which men will always give precisely the same amount of labour or of time. Hence there are hardly any two countries which use the same unit of value; and the unit itself varies much in any given country at different times. Gold and silver have been chosen in most civilized countries as the standards to which all other values are reduced. The reasons for this selection are—1. Because, compared with other commodities, their value fluctuates very little. 2. Because they are hard and durable materials. 3. Because they are capable of very accurate and minute subdivision. 4. Because, compared with other things of equal worth, they occupy very little space, and can easily be removed from place to place. Nevertheless, gold and silver, like other valuable things, rise and fall in price according to the quantity in the market; and even if they did not, we could not always measure the wealth of other times or other nations by a money standard. For in order to tell the worth of a given sum of money, we must know what amount of comforts or commodities can be purchased with it; and this varies very much under different circumstances.

The value of an ounce of gold* is £3 17s. 10½d. in this country, and this price has of late undergone very little fluctuation. The value of silver varies more, and is generally from 5s. to 5s. 6d. per ounce. It would therefore seem that the true way of comparing the value of foreign coinage with our own is to ascertain the fineness and weight of the metal of which it is made, and calculate its worth by this standard. But in fact the relative value of coinage in different countries is chiefly determined by the state of trade between them. For when two countries have commercial intercourse, money does

* This is not to be understood as perfectly pure gold; $\frac{1}{12}$ of it is supposed to be alloy, so that it would be more accurate to state that $\frac{11}{12}$ of an ounce, or 440 grains, represents the amount of pure gold in £3 17s. 10½d. An ounce of silver is supposed *to have* $\frac{3}{40}$, or 1½ dwt. alloy.

not actually pass from one to another, but bills are drawn (388) and are negotiated among the traders instead. Thus: if a French merchant has an English creditor, and an English merchant has a French creditor, it will not be necessary that each should ship gold or silver to pay his debts, if the former pays his neighbour the debt which he owes to the Englishman, and the latter pays to that Englishman the money due to his French correspondent. If all the Englishmen who owe money in France accept bills for the amounts, and all the Frenchmen who are indebted to persons in England do the same, these documents can be negotiated in the respective money markets, and no specie need be transmitted. Now if there be such an equilibrium in the trade of the two countries as that there is just the same sum owing by the whole body of English merchants that the whole body of French merchants owe to England, there will be as many bills drawn in England and payable in France as are drawn in France and payable in England. There will be just the same number of English debtors wishing to purchase bills upon France, as of English creditors wishing to dispose of them. The exchange is under such circumstances said to be *at par*. But suppose France is exporting more to us than we are sending to them. The sum of money owing by English merchants to France will be greater than that in which France is indebted to us. Hence in the French money market the supply of English bills payable to French merchants will be greater than the demand, and such bills will be sold at a discount, while in England, French bills will be at a premium, or will sell for rather more than they are worth. As in this case a balance of money is due to France, the exchange is said to be in favour of that country. The state of the market at any time is called the *course of exchange*. It is necessarily affected by other circumstances; the credit of the persons on whom the bills are drawn, the mint value of the specie in different countries, &c. But there is a limit to the fluctuations in the course of exchange. For though the transmission of specie involves inconvenience and risk, yet if the exchange were unusually to the advantage of France, a merchant in England would rather incur the expense of sending money to that country than purchase bills at a very high premium, and therefore at a loss.

Present English Money.

A farthing ($\frac{1}{4}$)	$= \cdot 001041\dot{6}$ of £1.		
1 penny (1d.)*	= 4 farthings	= £$\cdot 0041\dot{6}$	
1 shilling (1s.)†	= 12 pence	= 48 farthings	
1 sovereign or pound‡ (£1)	= 20 shillings	= 240 pence	= 960 farthings

* A penny weighs $10\frac{2}{3}$ drams (*avoir.*) or $291\frac{2}{3}$ grains of copper.

† A shilling weighs 3 dwt. $15\frac{3}{11}$ grains of silver.

‡ A sovereign weighs 5 dwt. $3\frac{171}{623}$ grains, or 123·274 grains of gold.

L, S, and D, are the initials of the words *Libri*, *Solidi*, *Denarii*, which are the

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TABLE OF FOREIGN MONEY.

	Sterling money.		Decimal of £1.	Value of £1 sterling at par.
	s.	*d.*		
FRANCE, BELGIUM, and SWITZERLAND.—1 *Franc* = 10 decimes = 100 centimes.		9¾	·039	25 fr. 22 cent.
UNITED STATES.—1 *Eagle* = 10 dollars: 1 *dollar** = 100 cents. .	4	6	·225	4 dol. 60 cents
AUSTRIA, BAVARIA, BADEN, and WIRTEMBURG.—1 *Florin* = 60 kreuzers.	2	0½	·102	9 fl. 3 kreuzers
PRUSSIA, SAXONY, and GERMAN ZOLLVEREIN.—1 *Thaler* = 30 silber groschen = 360 pfennings. .	2	10¾	·145	6 thalers 27 s. g.
FRANKFORT ON THE MAINE.—1 *Florin* = 60 kreuzers. . .	1	8	·083	12 florins
HAMBURG.—1 *Mark* = 16 schillings.	1	5½	·085	13 mks. 10 schs.
HOLLAND.—1 *Florin* = 20 stivers = 100 cents.	1	8	·083	12 fl. 9 cents
DENMARK.—1 *Rix-dollar* = 6 marks = 96 schillings. . . .	2	2½	·11	9 rix-dol. 10 sch.
†RUSSIA.—1 *Rouble* = 100 copecks.	3	2	·158	6 roubles 40 cts.
‡SPAIN, PERU, and BUENOS AYRES.—1 *Dollar of plate* or 1 *Piastre* = 8 reals = 272 maravedis. .	3	1	·154	6 dol. 4 reals plt.
1 *Hard dollar* = 20 reals (vellon).	4	2	·208	4 hd. dol. 16 rls.
PORTUGAL.—1 *Milrea* = 100 reas.	4	8½	·235	4 mil. 360 reas
SWEDEN and NORWAY.—1 *Rix-dollar* = 48 skillings.	1	8	·083	62 dol. 5 skils.
TURKEY and EGYPT.—1 *Piastre* = 40 paras.		2	·009	110 piastres.
GREECE.—1 *Drachma* = 100 centimes or ceptas.		8¾	·036	28 dr. 15 cept.
ROMAN STATES.—1 *Scudo* = 10 paoli = 100 bajocchi. . .	4	2	·208	4 scudi 6 paoli

plural forms of the names of three Roman coins. Until the alteration in the French coinage, the same words were employed in the form of *livres*, *sous*, and *deniers*.

* The actual value in silver of an American dollar is 4s. 1·2736d., but in exchange it is customary to consider 4s. 6d. as the standard, and to rate all fluctuations as so much above or below that sum.

† As the old style is still in use in Russia, 12 days must be added to the date of a Russian Bill of Exchange.

‡ In Spain two kinds of money are current, called respectively *plate* and *vellon; their values* being as 32 to 17. Thus: 17 reals plate = 32 reals vellon.

	Sterling money.		Decimal of £1.	Value of £1 sterling at par.
	s.	*d.*		
LOMBARDY.—1 *Lira* Austriacha = 100 centesimi.		8½	·034	29 liras 52 cent.
NAPLES and PALERMO.—1 *Ducat* = 5 tari = 10 carlini = 100 grani.	3	4	·166	6 ducats
SARDINIA and GENOA.—1 *Lira* = 100 centesimi.		9½	·04	25 liri 22 cent.
EAST INDIES.—1 *Rupee** = 16 annas = 192 pice = 5120 cowries. .	2	3	·112	8 ru. 24 ann.
CHINA and BURMAH.—1 *Tael* = 10 mace = 100 candarines . .	6	0	·3	3 tael 6 mace
MEXICO, CHILI, SINGAPORE, &c.—1 *Hard dollar* = 100 cents. .	4	2	·208	4 hard dol. 80 c.

PROPOSED ENGLISH MONEY ON A DECIMAL SYSTEM.

A mil = £·001; somewhat less than one farthing.
A cent = £·01 = nearly 2½d. of present money.
A florin = £·1 = 2 shillings.
A pound = 10 florins = 100 cents = 1000 mils.

MISCELLANEOUS AND OBSOLETE ENGLISH COINS.

A moidore = 27s.; a jacobus = 25s.; a carolus = 23s.; a guinea = 21s.; a mark = 13s. 4d.; an angel = 10s.; a noble = 6s. 8d.; a crown = 5s; a groat = 4d.

III. MEASURES OF LENGTH.

In early and rude calculations, the parts of the human body formed the principal standard of length. The terms foot, hand, palm, span, and pace, indicate the origin of our ordinary measures, and words equivalent to these are found in the languages of the Greeks and Romans. Among the Romans the *pes* or foot was the principal unit, and was equal to 11·6† modern English inches; the *uncia*, or inch, being $\frac{1}{12}$, or the breadth of the thumb, and the *digitus*, or finger breadth, being $\frac{1}{16}$ of a foot. The breadth of the four fingers (measured across the joints) gave the *palmus*, or hand, which was ¼ of the foot. The *cubitus* was the length from the elbow to the tips of the fingers, or 1 foot and a half. 2½ feet were one *gradus* or step; 2 steps, or 5 feet, one *passus* or pace; 1000 paces one *milliare* or

* A lac of rupees is 100,000, or £11,200.

† The Greek πους, or foot, was 12·14 English inches, or rather more than an English foot. This is ascertained by the measurement of some buildings still existing, whose dimensions are given by ancient writers. Yet the average length of a human foot is actually about 10·3 inches.

mile.* Of course it became necessary in very early times to fix an average standard for each of these lengths; and specimens of the legal lengths were probably kept in public buildings for reference and appeal.

The measurement in use in this country during the middle ages was founded on that of the Romans; but as the original standards were lost it became necessary to fix upon some others: grains of barley were employed for this purpose. The tradition is that at first four grains placed side by side were equivalent to the *digitus* or lowest measure.† By a statute of Edward II. it was enacted, that the length of three barleycorns, round and dry, taken from the middle of the ear, should make the legal inch, 12 inches one foot, 3 feet one yard, &c. It is commonly said that the legal measure of a yard was taken from the arm of Henry I.,‡ but this is very doubtful. Standards were made for reference and ordered to be kept in all large towns: these, however, were necessarily exposed to accident; and in fact the standard yard was destroyed in the fire at the Houses of Parliament, 1834. In ancient books of arithmetic also it was customary to illustrate the tables by printed lines or diagrams representing the various lengths; but this kind of standard is unsatisfactory, as paper is liable to shrink, and books are easily defaced or destroyed.

For all ordinary commercial purposes such standards as these are sufficiently accurate, the average length of barleycorns being probably uniform in successive years; but for scientific purposes it is desirable that the standard should be determined with mathematical exactness. It is for such purposes only that so many laborious investigations have been made with regard to the unit of length.

The great object therefore in regard to these measures, was to discover some one fact or object in nature which would furnish us with an unalterable standard of length. As physical science became more studied, the attention of learned men was directed to this subject. It was found that, in a given latitude, the pendulum which marked seconds was always of exactly the same length,§ and as the length of

* The assumption on which these measures are based, and which is on the whole an accurate one, is that in a man of average proportions, the breadth of the palm is $\frac{1}{24}$ of his height; the length of the foot one-sixth, and that of the cubit one-fourth.

† If this were true, 64 grains placed side by side would equal an English foot; but on putting this to a simple test, it may be found that 64 grains of even well grown barley give little more than $\frac{4}{5}$ of a foot.

‡ William of Malmesbury states positively that Henry I. commanded that the *ulna*, or ancient ell, which answers to the modern yard, should be made of the length of his own arm. But this is the only authority (and not a very trustworthy one) for the fact.

§ The Royal Commission of 1843 appointed to investigate this matter have however *reported*, that although the reference of the standard of length to the seconds *pendulum* was of great importance, on the whole they judged it better not to depend on

the seconds pendulum in London is 39·1393 inches, a ready means was discovered of correcting any future deviations from the true standard of length by referring it to the standard of time.*

It has now been shown—1st. That of all the magnitudes we wish to measure, *time* is that of which nature has most distinctly fixed the standard. 2ndly. That a method has been ascertained of determining the measure of length by that of time. 3rdly. That from a fixed length it is possible to deduce fixed measures of surface, of capacity, and of weight.

In the year 1826, an act of parliament was passed which settled the national usage in regard to weights and measures, abolished many that were cumbrous and inconvenient, and explained the connexion of the several tables with that called Long Measure, and of all of them with time. The following is the table in ordinary use:—

Long Measure.

1 inch†	= $\frac{1}{39\cdot13}$ of a seconds pendulum in the latitude of London				
1 foot	= 12 inches				
1 yard	= 3 feet	= 36 inches			
1 pole	= $5\frac{1}{2}$ yards	= $16\frac{1}{2}$ feet	= 198 inches		
1 furlong	= 40 poles	= 220 yards	= 660 feet	= 7920 in.	
1 mile	= 8 furlongs	= 320 poles	= 1760 yds.	= 5280 ft.	= 63360 in.

Cloth Measure.

1 nail	= $2\frac{1}{4}$ inches		
4 nails	= 1 quarter		
4 quarters	= 1 yard	= 16 nails	= 36 inches.

An English ell is 5 quarters, a French 6 quarters, and a Flemish 3 quarters.

Miscellaneous or Obsolete Measures of Length.

Hand (for measuring horses) = 4 inches; a *span* = 9 inches; a *cubit* = 18 inches; a *fathom* = 6 feet; a *chain* = 22 yards = 100 links; a *degree* = 69·1 miles; a *geographical mile* = $\frac{1}{60}$ of a degree; a Greek *stadium* = ·1149 English mile; a *parasang* = 30 stadia, or $5\frac{1}{2}$ English miles; a *log line*, used by sailors, = 48 feet; a *piece* of calico = 28 yards; a *piece* of Irish linen = 25 yards; a *piece* of muslin = 10 yards.

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such tests, but to cause a number of exact copies of a standard yard to be made and deposited in secure places throughout the country. Gun metal, composed of copper, tin, and zinc, in the proportion of 16, $2\frac{1}{2}$, and 1, was recommended as the best material, being hard, elastic, and less affected by change of temperature than others.

* Not that a second is a period of time which nature measures for us, but because it is an aliquot part of the natural period called a day.

† We have no names for the subdivision of an inch, which may be eighths, twelfths, or tenths; barleycorns are entirely out of use.

French Linear Measures.

The entire system of weighing and measuring in France was remodelled at the end of the last century, with a view to the reduction of all tables to a decimal form. The plan adopted was to fix on one definite length, one surface, one cube, and one weight (the last three being founded on the first), and to employ the same syllables in each table to express the measures and multiples of these units. So, for the various decimal multipliers, terms were taken from the Greek language, while for the decimal divisors the syllables were chosen from the Latin.

e.g., *Myria* (μυρια) = 10000. *Kilo* (χιλια) = 1000. *Hecto* (ἑκατον) = 100. *Deca* (δεκα) = 10. *Deci* (decem) = one-tenth. *Centi* (centum) one-hundredth. *Milli* (mille) = one-thousandth.

The unit of length chosen is the ten-millionth part of the distance from the equator to the pole; it is called a *metre*, and is equal to 39·371 English inches.* The table is as follows :—

Myria-metre	= 10000 metres	=	393710 =	English inches
Kilo-metre†	= 1000 metres	=	39371	"
Hecto-metre	= 100 metres	=	3937·1	"
Deca-metre	= 10 metres	=	393·71	"
Metre, principal unit		=	39·371	"
Decimetre	= $\frac{1}{10}$ of a metre	=	3·9371	"
Centimetre	= $\frac{1}{100}$ of a metre	=	·39371	"
Millemetre	= $\frac{1}{1000}$ of a metre	=	·039371	"

IV. Measures of Area or Surface.

These measures are obviously dependent on the preceding. A square surface having a given unit of length for its base being a good measure of the area.

Land or Square Measure.

1 square inch	= a square surface having a linear inch for [each of its sides
1 square foot	= 144 sq. inches
1 square yard	= 9 sq. feet = 1296 square inches
1 square pole	= $30\frac{1}{4}$ sq. yards
1 square rood or perch	= 40 sq. poles = 1210 sq. yds.
1 acre	= 4 sq. roods = 4840 sq. yds. = 100000 [sq. links
1 square mile	= 640 acres

Miscellaneous or Obsolete Measures of Surface.

A *hide* of land = 100 acres; a *yard* of land = 30 acres; a *square of flooring* = 100 square feet; 1 *rod of brickwork* = $272\frac{1}{4}$ square feet.

1 quire of paper = 24 sheets; 1 ream = 20 quires; a printer's ream = $21\frac{1}{2}$ quires.

* Nearly the same length as the seconds pendulum.

† This is the measure most frequently quoted in France for long distances, and it is important to remember that it is about 5 furlongs.

FRENCH SUPERFICIAL MEASURE.

The principal unit chosen is the *are* or square decametre, *i.e.*, a square having 10 *metres* or linear units for one of its sides.

Hectare*	=	11960·46
Decare	=	1196·046
Are, principal unit	=	119·6046 square yards English
Deciare	=	11·96046
Centiare	=	1·196046

V. MEASURES OF SOLIDITY OR CAPACITY.

These may easily be deduced from those of surface and of length, for a solid in a cubical shape and having a given unit of length for one of its edges, is the only standard we need either to measure the bulk of solids or the capacity of vessels. Nevertheless the old English measures were not formed in this way, but by a less direct method. A statute of Henry III. (1266) enacts that 32 grains of wheat well dried shall be the legal weight of a silver penny, and shall be called a penny-weight; 20 such pennyweights shall make an ounce, 12 ounces a pound, and 8 pounds of wheat thus counted and weighed should fill a gallon. Thus the gallon became the standard unit of capacity, and was derived from weight, and not immediately from length. But by the act of 1826 it was ordered that the gallon should contain exactly 277·274 cubic inches. Before this act the gallon used to measure corn was less than that for wine, as 268·6 to 282. The *shape* of the measuring vessel was fixed by act of parliament, and it was further directed that the articles thus measured were to be heaped above the rim of the vessel in the form of a cone, to one-third of the height of the measure. Now that heaped measure is abolished and the exact number of cubic inches fixed, it is unnecessary to insist upon any particular form of the vessel.

The *gallon* is the standard measure of capacity both for dry goods (corn, &c.) and for liquids. Special names are employed in Wine, Ale, and Beer Measures, but these are rather names of casks than standard measures, for by the act of William IV. the contents are always to be guaged in gallons.

SOLID MEASURE.

1 cubic foot = 1728 cubic inches
1 cubic yard = 27 cubic feet

* Large surfaces are generally spoken of in France as *Hectares*, each of which is about 2½ acres English.

Imperial Measure.

A gill	= $\frac{1}{4}$ of a pint	= $\frac{1}{32}$ of a gallon
A pint		= $\frac{1}{8}$ of a gallon
A quart*	= 2 pints	= $\frac{1}{4}$ of a gallon
A *gallon*†		= 277·274 cubic inches
A peck	= 2 gallons	= 8 quarts
A bushel	= 4 pecks	= 8 gallons
A quarter	= 8 bushels	= 64 gallons
A load	= 5 quarters	= 320 gallons

Wine Measure.

1 anker	= 10 gallons
1 runlet	= 18 gallons
1 tierce	= 42 gallons
1 puncheon	= 84 gallons
1 hogshead	= 63 gallons
1 pipe	= 126 gallons
1 tun	= 252 gallons

Ale and Beer Measure.

1 firkin	= 9 gallons
1 kilderkin	= 18 gallons
1 barrel	= 36 gallons
1 hogshead	= 54 gallons
1 butt	= 108 gallons
1 tun	= 216 gallons

Miscellaneous Measures of Capacity.

1 sack = 3 bushels; 1 chaldron = 36 bushels; 1 *load* or *ton of hewn timber* = 50 cubic feet; 1 *load of rough timber* = 40 cubic feet; 1 *ton* (in shipping) = 42 cubic feet; 1 *ton* of marble = 12 cubic feet; 1 *ton* of Portland stone = 16 cubic feet; 1 *ton* of Bath stone = 20 cubic feet.

French Measures of Capacity.

The unit employed for the measurement of corn and dry goods, as well as liquids, is the cube of the *decimetre*. This is nearly equal to our pint, and contains 61·028 cubic inches.

Hectolitre	= 6102·8	cubic inches
Decalitre	= 610·28	cubic inches
Litre, principal unit	= 61·028	cubic inches
Decilitre	= 6·1028	cubic inches
Centilitre	= ·61028	cubic inches

The unit employed for solidity is the *cube of the metre*, and is called a *stere;* it is 35·317 cubic feet.

VI. Measures of Weight.

It has been seen that at one time grains of barley, carefully numbered or measured, formed the standard of weight. From that it would appear that 7680, or 32 × 20 × 12, was the number of grains in an old pound. But as this mode of weighing is liable to

* *Quart*, from *quartus*, = one-fourth.

† *The imperial* gallon contains about 10 lbs. avoirdupois of pure water.

variation, a method was sought by which the unit of weight might be referred to that of capacity. Now water, at a given temperature and under a given barometic pressure, never varies in specific gravity, and a cubic foot, or 1728 cubic inches of water, at a temperature of 62 degrees, and when the barometer is at 30 inches, is found to weigh nearly 1000* ounces avoirdupois, or 62·32106 pounds. Hence the unit of capacity becomes a measure of weight. But this is not considered perfectly satisfactory, and it was recommended by the commissioners who reported to parliament on this subject in 1843, that standards of metal should be made and kept by the nation, in preference to deducing the unit of weight from that of length.

The weight now called troy weight is undoubtedly of more ancient use than that known as avoirdupois. The troy pound consists of 5760 grains (for though 32 grains originally made a pennyweight, 24 was the number even before the time of Henry VII.) This king is said to have fixed the troy pound at its present weight, and to have increased the old Saxon pound by three-quarters of an ounce. By the statute of 1826 the use of this weight is limited to gold, silver, platina, diamonds, precious stones, and such drugs as are sold by retail. Until the time of Henry VIII. gold and silver were weighed by the Tower pound, which was 11¼ ounces.

There was a weight in use in business up to this period called the merchant's pound, of 15 ounces troy. This is plausibly conjectured to be the origin of the modern avoirdupois pound, which is 14·6 ounces troy. There is good reason to believe that when the word avoirdupois first became in use it was applied to this greater pound. The present standard of the avoirdupois pound was fixed in the reign of Elizabeth. It contains 7000 troy grains. It therefore is to a troy pound as 7000 is to 5760.

The great or merchant's pound consisted of 7680 grains, and is the same as the old pound just described, which had 32 grains to a pennyweight; of these grains 20 made 1 scruple, 3 scruples a drachm, 8 drachms an ounce, 16 ounces 1 pound. It appears that long after the settlement of the avoirdupois standard medicines continued to be dispensed by this ancient sub-division of the pound. Now these grains, scruples, and drachms, are not aliquot parts of an avoirdupois pound, or of 7000 grains, but they are measures of the troy pound. And thus it happens, that while apothecaries buy and sell drugs wholesale by the avoirdupois pound they retail them by the troy pound, only sub-dividing it in a different way. Hence—

$$1 \text{ lb.} = \begin{cases} 12 \text{ ounces} = 240 \text{ dwts.} = 5760 \text{ grs. (troy)} \\ 12 \text{ ounces} = 96 \text{ drachms} = 288 \text{ scr.} = 5760 \text{ grs. (apoth.)} \end{cases}$$

Hence a troy ounce : avoirdupoise ounce :: $\frac{5760}{12} : \frac{7000}{16} = 96 : 85$.

* The exact number of ounces is 997·14.

Troy Weight.

1 pennyweight	= 24 grains		
1 ounce	= 20 dwt.	= 480 grains	
1 pound	= 12 oz.	= 240 dwt.	= 5760 grains

Apothecaries Weight.

1 scruple (℈)	= 20 grains			
1 drachm (ʒ)	= 3 scruples	= 60 grains		
1 ounce (℥)	= 8 drachms	= 24 scruples	= 480 grains	
1 lb.	= 12 ounces	= 96 drachms	= 288 scr.	= 5760 grs.

Avoirdupois Weight.

1 dram				
1 ounce	= 16 drams			
1 lb.	= 16 ounces	= 256 drams		
1 quarter	= 28 lbs.	= 448 ounces	= 7168 drams	
1 cwt.	= 4 quarters	= 112 lbs.	= 1792 oz.	= 28762 drs.
1 ton	= 20 cwt.	= 80 quarters	= 2240 lbs.	= 35840 oz.

Miscellaneous and Obsolete Measures of Weight.

A stone = 14 lbs. (avoirdupois); a stone of meat = 8 lbs.; a sack of coals = 2 cwt.; a *truss* of *straw* = 36 lbs.; a truss of hay = 60 lbs.; a load = 36 trusses; a pack of wool = 240 lbs.; 1 *firkin of butter* = 56 lbs.; 1 *fother of lead* = 19½ cwt.; a *stone* of glass = 5 lbs.; a *pocket* of hops = 112 lbs.; a *firkin* of soap = 64 lbs.; a *gallon* of salt = 7 lbs.; a *bag* of rice = 168 lbs.; a *chest* of tea = 84 lbs.; a *quintal* of fish = 112 lbs.; a *barrel* of anchovies = 30 lbs.; a *barrel* of flour = 196 lbs.; a *gallon* of oil = 9 lbs.

Wool Weight.

1 clove	= 7 lbs. (avoirdupois)
1 stone	= 2 cloves
1 tod	= 2 stones
1 wey	= 6½ tods
1 sack	= 2 weys
1 last	= 12 sacks

French Measures of Weight.

The unit of weight is derived from that of capacity; a cubic centimetre of distilled water, at its maximum density, being taken as the standard. This weight is called a *gramme*.

Kilo-gramme*	= 15444·40234 grains
Hecto-gramme	= 1544·4023 grains
Deca-gramme	= 154·4402 grains
Gramme, principal unit	= 15·4440 grains
Deci-gramme	= 1·5444 grains

* As the kilogramme is the weight most frequently quoted, the student should *carefully* notice its magnitude, which is about 2¼ pounds avoirdupois.

VII. Measures of Time.

We have seen that a day is the universal unit employed for measuring time. However varied the mode of reckoning smaller periods may be among different nations, there can be no difficulty in comparing them, as they are all aliquot parts of a day. The only difficulty attending the construction of these tables is, that there are longer periods of time indicated by nature, such as that of the revolution of the moon round the earth, and of the revolution of the earth round the sun; and these two periods, *a month and a year, are incommensurable with a day and with each other.* We can neither express a year nor a lunar month exactly by days and fractions of a day. The solar year is 365·2422414 days nearly, and this fraction cannot be expressed by an exact number of hours, minutes, or even seconds. Again: the period of one lunation, or the time between two new moons, is 29·5305887 days; and this fraction is likewise incapable of being expressed in minutes and seconds.

Though the determination of a day is very simple, it required very careful observation to register the exact length of the solar year. Hence among semi-barbarous nations it was commonly the case that a number of days either too great or too small was fixed upon to represent the year. Thus, the Egyptians reckoned the year of 360 days. Numa, the second king of Rome, formed the year of 12 lunar months (of 29 and 30 days alternately), or 354 days. To make this arrangement agree with the actual solar year, an intercalary, or extra month, was introduced every two years. But as the length of this extra month, and even the periods for its occurrence, were left to the discretion of the pontiffs, great irregularities occurred. In the later time of the republic the additional month was not inserted so often as was necessary, and the average year of the Romans was considerably less than the solar year. The effect of this was to make the artificial computation in advance of the real time; and when the error had accumulated to rather more than two months, the Romans were, in fact, calling the beginning of June that which should have been the end of March. To remedy this, Julius Cæsar, in the year 47 B.C., caused the year to consist of 445 days, thus adding 67 days to the ordinary calculations. The effect of this was to bring the nominal year into harmony with the natural one. At the same time, with the aid of some Egyptian astronomers, he ascertained pretty nearly the exact length of the solar year. He took it to consist of 365¼ days, and enacted that every fourth year there should be an additional day to make up for the four quarters. As this additional day was added by causing the sixth of the calends of March (23rd of February) to be repeated twice, the year in which the day was added was called bissextilis (*bis* = twice, *sextilis* = sixth), or the year in which there were two sixths of March.

The Julian calendar thus fixed was adopted throughout Europe But it was based on the supposition that the year contained 365

days 6 hours (365·25 days), whereas the true year only contains 365 days, 5 hours, 48 minutes, 49·7 seconds (365·2422414): an error of little more than 11 minutes per annum was thus caused. This error it should be observed was an error of the opposite kind to that which the Romans had made; for whereas, by making their year too short, their calculations were in advance of the real year, the Julian calendar, by making each artificial year 11 minutes longer than the real one, allowed the true year to be in advance of the ordinary calculation. In the sixteenth century it was observed that the error thus caused had accumulated to 10 days. Men were calling that the 10th of March which ought to have been called the 21st. Pope Gregory XIII., in 1582, proposed to correct this by omitting 10 days from the nominal year; and with a view to prevent the recurrence of the mistake, he enacted that the extra day should be omitted in every 100th year, except the 400th. Thus of the extra days fixed by the Julian calendar 3 are to be omitted in every 400 years. By this arrangement, though it is not a perfect one, it is calculated that the error in computation will not amount to 1 day in 6000 years.* It is important to remember that as this arrangement of time, called the *new style*, did not take effect in this country until 1751, when the error had amounted to 11 days, all earlier dates which occur in history, or in old documents, require to be corrected by the addition of 11 days. So that the years 1700, 1800, and 1900, were not to be leap years, but 2000, 2400, 2800, are to have a 29th of February as usual.

TIME TABLE.

1 minute = 60 seconds
1 hour = 60 minutes
1 day = 24 hours
1 week = 7 days

The year is rather irregularly divided into 12 parts, averaging 30·416 days each. Each of these is called a *calendar* month: 28 days, or 4 weeks, are called a *lunar* month.

MISCELLANEOUS MEASURES OF TIME.

1 lustrum = 5 years; 1 century = 100 years; 1 lunar cycle = 19 years; 1 solar cycle = 28 years.

* It is interesting to observe how very reluctantly this arrangement was adopted by Protestant countries. The change took effect in France, Spain and Portugal, and Italy, immediately on its promulgation by the Pope in 1582. The new style was legally established throughout the Netherlands in the same year; but several of the provinces refused to adopt it, and in Utrecht and Guelders it was not in use until 1700. Throughout Germany and Switzerland the Catholics received it in 1584, the Protestants not until 1699. In Sweden the new calendar commenced in 1753. In our country the change was resisted until 1751, and was very unpopular. To this day Russia and the countries in connexion with the Greek church adhere to the old style.

APPENDIX.—B.

VARIOUS SCALES OF NOTATION.

It has been shown that because we have taken ten for the base of our notation—I. That we need distinct significant characters for the first *nine* numbers only and express all higher numbers by varying their positions; and II. That every higher number than 9 has one part which is considered as a product, having for one of its factors *ten*, or some power of ten. Thus—

$$22222 = 2(10^4) + 2(10^3) + 2(10^2) + 2(10) + 2.$$

Now it follows from similar considerations, that if 4 had been chosen for the base of our system we should only have needed *three* significant figures; 10 would have meant *four*, and 22222 would have equalled $2(4^4) + 2(4^3) + 2(4^2) + 2(4) + 2$.

And because that which we now call 37, or $(3 \times 10) + 7 = (2 \times 16) + 4 + 1$, or $2(4^2) + 4 + 1$;

Therefore 37, on the decimal scale, might have been expressed as 211 on the quaternary scale, or as $(2 \times 4^2 + 4 + 1)$.

And whereas in considering ordinary numbers we break them up into tens, hundreds, and thousands, it would be necessary, if 4 were the base of our notation, to decompose every number into fours and powers of four.

Suppose it is required to find how 500 would have been written had either 4 or 7 been the base of our notation; it only becomes necessary to divide it into fours and sevens in the following manner:—

$$\begin{array}{r} 4)\underline{500} \\ 4)\underline{125\cdot 0} \\ 4)\underline{31\cdot 1} \\ 4)\underline{\ 7\cdot 3} \\ \underline{1\cdot 3} \\ 13310 \end{array} \qquad\qquad \begin{array}{r} 7)\underline{500} \\ 7)\underline{71\cdot 3} \\ 7)\underline{10\cdot 1} \\ \underline{1\cdot 3} \\ 1313 \end{array}$$

13310 on the quaternary scale $=$ 500 on the decimal; for

$$\begin{array}{rl} \underline{13310} & \\ 10 = & \ \ 4 \\ 300 = & 3 \times 4^2 = \ \ 48 \\ 3000 = & 3 \times 4^3 = 192 \\ 10000 = & 4^4 = \underline{256} \\ & \ 500 \end{array}$$

1313 on the septenary scale $=$ 500 on the decimal; for

$$\begin{array}{rl} \underline{1313} & \\ 3 = & 3 \\ 10 = & 7 \\ 300 = & 3 \times 7^2 = 147 \\ 1000 = & 7^3 = \underline{343} \\ & \ 500 \end{array}$$

Whatever number be chosen as the base, one less than that number of digits would be required. Hence if eleven, twelve, or any

greater number than ten were chosen it would be necessary to have new characters; suppose twelve were the base, 10 would mean twelve, x might mean ten, and e eleven; then $25e7t$ would equal $(2 \times 12^4) + (5 \times 12^3) + (11 \times 12^2) + (7 \times 12) + 10$.

The following examples will serve to show how any one of the four fundamental operations of arithmetic might be performed on any scale which might be chosen.

I. Find 13 + 213 + 402 + 12 on the quinary (5) scale.

Scale of five.		Scale of ten.
13	=	8
213	=	58
402	=	102
12	=	7
2100	=	175

II. Subtract $142x$ from $27e9$ on the duodecimal scale.

Duodecimal.		Decimal.
$27e9$	=	4605
$142x$	=	2338
$138e$	=	2267

III. Multiply 12212 by 7 on the ternary scale.

Ternary.		Decimal.
12212	=	158
21		7
12212		
102201		
1111222	=	1106

IV. Divide 2648 by 7 on the nonary scale.

7)2648	=	7)1988
345	=	284

EXERCISE.

Convert 52364 into the scale whose base is 5.

How would 10000 be expressed if the radix were 2, 7, or 11?

What number in the decimal scale equals 123421 on the quinary?

If 23454 be a number on the septenary scale, how will it be expressed on the scale whose radix is 8?

Add together 123, 432, 310, and 212 on the scale whose base is 6; multiply the result by 14, and give the answer in the duodecimal scale.

MISCELLANEOUS QUESTIONS.

How many pieces of cloth 9 yards 2 qrs. 3 nails long can be cut out of a piece 52 yards 1 qr. 1 nail in length?

Find the value of 227 qrs. 3 bush. 2 pecks of wheat, at 36s. 8d. per quarter.

How many ounces of silver, at 5s. 6d. per ounce, are equivalent to 6 oz. 12 dwt. of gold at £3 17s. $10\frac{1}{2}$d. per oz.

The sun's diameter is 111·454 times the equatorial diameter of the earth, which is 7925·648 miles. Required the sun's diameter in miles?

A manufacturer having a capital of £5000, on which he can realize by hand labour 10 per cent. profit, buys a machine for £1000, by which the profit on the remainder of his capital is raised to 20 per cent. This machine lasts 5 years; how much is he by that time the gainer, supposing him to draw £300 a year for the support of his family?

Find the value of—

(a) 4 cwt. 2 qrs. 16 lbs. at £3 17s. $8\frac{3}{4}$d. per cwt.

(b) 3 oz. 17 dwt. 20 grs. at 8s. $6\frac{1}{4}$d. per oz.

In the Julian correction of the calendar, every fourth year (leap year) consists of 366 days; in the Gregorian correction leap year is omitted three times in four centuries, but otherwise retained. Compare the mean lengths of the year according to those corrections respectively with the true length, which is 365 days 5 hours 48 min. 49·7 sec. nearly.

By the use of the Julian correction, how many superfluous days would have been introduced from the year 1 to the year 1750 A.D.?

Having bought goods for £20, I sell half of them so as to gain 10 per cent.; for how much must I sell the remainder so as to gain 20 per cent. on the whole?

Having bought goods for £18, I sell them four months afterwards for £25; what is the gain per cent. per annum?

Find the area of a floor whose length is 13 ft. 7 in. and its breadth 11 ft. 9 in., and show by means of a diagram what surface is represented in each term of the answer.

Between 1801 and 1811 the population of Edinburgh increased by $24\frac{3}{4}$ per cent., and in the latter year it was 102987. What was it in 1801?

Assuming that there are 58240000 acres of land in the United Kingdom, how many square miles of land does it contain?

Show that any whole number, 765468, may be expressed by the sum of its digits multiplied by powers of 10. Prove also by general reasoning that a number is divisible by 9 when the sum of its digits is divisible by 9, and only then.

Add together the fractions $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{11}{12}$, $\frac{19}{36}$, and explain why they

must be first reduced to a common denominator. What fraction must $\frac{2}{3}$ be divided by to give a quotient $\frac{11}{12}$? Can more than one such fraction be found?

What fraction of the earth's diameter (7900 miles) is a mountain $4\frac{1}{2}$ miles high? By what fraction of an inch would the height of such a mountain be properly represented on a globe of 18 inches diameter?

If the whole revenue of the country (£50,000,000) were paid as interest on the national debt, £760,000,000, how much per cent. would it give?

Find the interest on £7650 10s. for 5 years, at $3\frac{1}{2}$ per cent.

Multiply 2·564 by ·047, and divide ·00169 by ·013; verify your results by putting the decimals in the form of vulgar fractions.

Reduce $\frac{17}{625}$ to a decimal, and explain why, in reducing a fraction to a decimal which terminates, the number of decimal places depends on the form of the denominator of the fraction, and not on that of the numerator. Extract the square root of 258368·89.

When are magnitudes (1) in arithmetical (2) in geometrical progression? Find an arithmetical and a geometrical mean between (a) and (b), write down the 12th term of the series 7, 12, 17, and find the sum of five terms of the series $\frac{1}{3} + \frac{2}{9} + \frac{4}{27}$, &c.

If 9000 persons, travelling each 20 miles weekly, give a railroad company a receipt of £900 in one week, how many persons travelling each 30 miles weekly will give a receipt of £62400 a year, when the cost of travelling per mile is reduced one-half?

If the 8d. loaf weighs 48 oz. when wheat is at 54s. per quarter, what should be the price of wheat when the 6d. loaf weighs 32 oz. 8 dwt.?

How many yards of carpet, 2 ft. 11 in. wide, will it take to cover a square floor, one side of which is 19 ft. 7 in.?

A tradesman having bought 200 eggs at 2 for a penny and 200 at 3 for a penny, sold the whole at 5 for a penny; how much did he gain or lose by the transaction?

Find the interest on £286 from the 1st of June to the 15th of September, at $3\frac{1}{2}$ per cent. per annum.

A, B, and C are joint owners of a ship; C's share is worth £400, A's share is $\frac{1}{3}$ of B's, and the sum of their shares is six-eighths of the value of the ship. Find the value of the shares held by A and B.

A person who has two-fifths of a mine sells three-fourths of his share for £1500; what is his share in the whole mine?

Reduce 1 cwt. 3 qrs. 5 lbs. to the decimal of $\frac{2}{5}$ of a ton.

I borrow £130 on 5th March, and pay back £132 10s. 6d. on 18th October; what rate per cent. per annum of interest have I paid?

A and B rent a field for £35 a year; A puts in six horses for the whole year, B puts in five horses for eleven months, and three more for five months; how much should each contribute towards the rent?

I wish to measure a distance of three furlongs with a line three

rods and a half in length; how many times will the line measure the distance?

Add together the circulating decimals 0·53434, &c., and 0·465858, &c., and subtract the sum from $1\frac{1}{3}$.

Perform the operations indicated below :—

(1) $36{\cdot}01 - 2{\cdot}987564$ (4) $6{\cdot}25 \div {\cdot}000125$

(2) $2{\cdot}745 \times 45{\cdot}674$ (5) $\sqrt{2119{\cdot}6816}$

(3) $233{\cdot}8268 \div 3{\cdot}46$

Find the value of $\cdot 3333\dot{3}$ of $2\frac{1}{4}$ guineas.

Sum the series $4 + 11 + 18 + \cdots$ to 9 terms,

And also the series $3 + 6 + 12 + \cdots$ to 16 terms.

If 15 men, 12 women, and 9 boys can complete a piece of work in 50 days, how long would 9 men, 15 women, and 18 boys be in doing double the work; the parts done by each man, woman, and boy respectively, in the same time, being as the numbers 3, 2, 1?

Given that the French metre is equal to 39·371 English inches, and that 100 Hamburgh miles are equal to 468·41 English miles; express one mile of Hamburgh in terms of French metres.

A privateer running at the rate of 10 miles an hour discovers a ship 18 miles off, making way at the rate of 8 miles an hour; how many miles can the ship run before she will be overtaken?

If 12 men build a wall 60 ft. long, 4 ft. thick, and 20 ft. high, in 24 days, working 12 hours a day, how many must be employed to build a wall 100 ft. long, 3 ft. thick, and 12 ft. high, in 18 days, working 8 hours a day?

A and B exchange goods; A gives 13 cwt. of hops, the retail price of which is 56s., but in barter he rates them at £3; B gives A 10 barrels of beer, the retail value of which is 1s. a gallon, but the value of which he raises in proportion to the increased value of the hops; how much must B give in money?

Find the sum of $\frac{3}{7}$ of a guinea, $\frac{3}{8}$ of a pound, and $\frac{7}{20}$ of a crown.

Place the first term in the following proportion :—

$$x : (15 + 9 - 2)^2 :: 5 \times 3 \times 2^4 : 6^2 + (6 \times 2^7).$$

If a spoon weigh 15 dwt. 11 grs., how many dozen of such spoons can be formed out of 122 oz. 9 dwt. 1 gr.?

A field of 16 acres produces 440 bushels of wheat; how much is that upon every 22 square yards?

A man spends 12 guineas in 35 days, and saves £100 a year; what must he earn in the year?

Suppose 8 men can do the work of 35 children, and 12 women the work of 19 children, how many children can do the work of 17 men?

Required the number of square feet there are in a piece of slate $2\frac{1}{4}$ ft. and $\frac{1}{3}$ in. long, and $1\frac{1}{3}$ ft. and $\frac{1}{2}$ in. in width.

What is the value of the flooring of a schoolroom consisting of 36 planks, each plank $10\frac{1}{2}$ ft. long, 8 in. wide, and 3 in. thick, a cubic foot being worth 1s. $7\frac{1}{2}$d.?

There are 10 windows in a house; each window contains 12 panes of glass; what is the value of the whole glass, each pane being $1\frac{1}{3}$ ft. long, 10 in. wide, and $\frac{1}{5}$ in. thick, a cubic inch being worth $2\frac{1}{4}$d.?

A boy loses $\frac{1}{3}$ of his marbles; he plays again and wins $\frac{4}{7}$ of $\frac{3}{4}$ of what he has left; he now has 80 marbles: how many did he have at first?

If $\frac{4}{5}$ of the rent of a house be £32 7s. $8\frac{1}{2}$d., what is the rent?

Divide £15 4s. 6d. among 4 men, 5 women, and 6 children; give the men each a share, the women each $\frac{3}{4}$ of a share, and the children each $\frac{2}{5}$ of a share.

Solve the following expressions:—

$$\frac{\dfrac{2\frac{1}{2} \text{ of } 1\frac{1}{4}}{3\frac{2}{5} \div 7\frac{5}{8}}}{\dfrac{4\frac{4}{9} \text{ of } 6\frac{1}{27}}{1\frac{1}{18} + 4\frac{4}{9}}} \times \frac{3\frac{4}{7}}{3\frac{1}{14}} \quad \text{and} \quad \frac{\dfrac{6 - \frac{2}{7}}{11 + \frac{3}{7}} \times \dfrac{6\frac{1}{8} \div \frac{147}{575}}{19\frac{1}{9} \times \frac{1}{11}}}{\dfrac{\frac{19}{20} \times \frac{3}{7} \text{ of } \frac{14}{19}}{\frac{5}{16} \text{ of } \frac{4}{11} \div \frac{8}{19}} \div \dfrac{\frac{7}{12} \times \frac{3}{5}}{\frac{11}{8} + \frac{4}{13}}}$$

If the sun shining on the ocean 7 hours per day causes 8756·75 cubic feet of water to evaporate per hour on a surface of 2·875 square miles, what would be the weight of water evaporated on a surface of 879·398 square miles in 6 days $2\frac{1}{2}$ hours, supposing the sun to shine 9 hours 37 minutes per day, and the weight of a cubic foot of water to be $62\frac{1}{2}$ lbs.?

If 834 horses can plough $6756\frac{108}{199}$ acres in $7\frac{228}{347}$ days, working $7\frac{5}{8}$ hours per day, what must be the length of the working day when 946 horses plough $6729\frac{566}{597}$ acres in $6\frac{547}{963}$ days, supposing each horse in the latter case to perform only $\frac{11}{12}$ of the work performed by each in the former?

A stage coach has the circumference of its fore wheels $4\frac{1}{8}$ ft., and the circumference of the hind ones $12\frac{4}{19}$ ft. It is required—1. To find the number of revolutions performed by the fore and hind wheels in travelling $\frac{1}{7}$ of $\frac{1}{18}$ of the circumference of the earth (taken equal to 24900 miles). 2. If the hind wheels make 5 revolutions every two seconds, how many revolutions do the fore ones perform per second? 3. In what time will the distance be travelled over according to this rate?

A colonist procures 9000 acres of land from Government—$\frac{1}{5}$ of which is to be grazing land, $\frac{3}{7}$ arable land, and the remainder for the produce of hay—upon the following conditions; viz., that he is to give to the Government $\frac{3}{5}$ the annual profit of the grazing land, $\frac{1}{4}$ the annual profit of the arable land, and $\frac{1}{2}$ of the third kind of land. Now the whole annual profit from the first is £1 per acre, from the second £$1\frac{2}{5}$ per acre, and from the third £$\frac{9}{13}$ per acre; required the whole annual profit of the farm, and the sum due to each of the parties.

A ship at sea is known to sail at the rate of 10 miles per hour when the tide is with her; on the tide returning, her rate of *sailing is* reduced $\frac{1}{3}$ the former rate; after sailing for some time at

this rate, the wind increasing, her speed is increased $\frac{1}{5}$ the last rate; required the distance travelled over in 12 hours, supposing her to sail $\frac{1}{5}$ of the time as in the first case, $\frac{2}{3}$ of the time as in the second case, and the remainder as in the third case.

Express the answer to the above question in French metres, each 39·371 inches, and also in a standard of measure equal to the length of a pendulum vibrating seconds in the latitude of London, which is 39·1393 inches.

If I transfer £7280 from one kind of stock which is at 69·25, and pays 3 per cent., to another kind of stock at £108·6, which pays 5 per cent., what will be the difference in the annual income arising from the investment?

If I have to pay a bill of £370 at 3 months' date from this time, and I pay £120 of this sum at once, what extension of time ought to be allowed for the payment of the remainder?

Find the least number which is divisible by 9 without a remainder, and which, when divided by 7, leaves a remainder 4.

Two detachments of foot being ordered to a station at the distance of 39 miles from their present quarters, began their march at the same time; but one party, by travelling $\frac{1}{4}$ of a mile an hour faster than the other, arrived there an hour sooner; required their rates of marching.

Having obtained the five first digits of the decimal equivalent to $\frac{1}{17}$ by actual division, deduce from the result the complete recurring period for $\frac{6}{17}$.

How much stock can be purchased by the transfer of £2000 stock from the 3 per cents., at 90, to the $3\frac{1}{2}$ per cents., at 96; and what change will be effected in income by it?

A ship having a crew of 26 persons, carries provisions for 21 days; after having been at sea for 11 days, they pick up a party from a wreck, and it is then found that the provisions will be exhausted in the course of 5 days; find the number of persons taken from the wreck.

If the 3 per cent. stock be at 98, and the $3\frac{1}{4}$ per cent. stock be at 101, which stock is it most advantageous to buy? What will be the difference between the annual income derived from £5000 invested in the two ways?

I paid £28 per cwt. for tea, and sold it again for £32 10s. per cwt.; what quantity must be sold to gain £169 17s. 6d.?

A lump of iron, containing 11 cubic feet, is drawn out into a rod 14 yards long; what will be the thickness of the rod?

A contract is to be finished in 5 months 17 days, and 43 men are put on to work at once: at the end of $\frac{2}{5}$ of this time, it is found that only $\frac{1}{7}$ of the work is done; what extra number of hands will be required to complete the contract in the given time, the last employed men to work 12 hours per day, whilst the first 43 men work until the contract is completed only 10 hours per day?

ANSWERS TO THE EXERCISES.

EXERCISE III.

Two thousand three hundred and five; eight hundred and six; seven thousand and ninety-five; twenty thousand three hundred; four hundred and fifty-seven thousand two hundred and ninety-eight; six hundred and twenty-seven thousand four hundred and twenty-one; thirty-three thousand nine hundred and eleven; four hundred and twenty-seven thousand eight hundred and sixteen; nine millions and thirty-two thousand eight hundred and four; eight million two hundred and seventy-one thousand and ninety-six; thirty-two million seven hundred and forty-five thousand eight hundred and forty-one; seventy-two thousand nine hundred and eighteen; thirteen; five thousand one hundred and seventy-two; eight hundred and forty; six hundred and twenty-one.

V.—960; 40002; 7600; 81402; 250; 5004007; 8600; 24005; 1215; 6110; 981; 18000006; 413; 500.

VI.—(b) 5000 + 1100 + 12 tens + 14; 4000 + 1800 + 17 tens + 17; 3000 + 1100 + 15 tens + 11; 40 thousands + 13 thousands + 1700 + 11 tens + 13; 1000 + 900 + 18 tens + 16; 70000 + 12000 + 1300 + 11 tens + 17; 1800 + 15 tens + 14; 200 + 17 tens + 17; 500000 + 90000 + 18000 + 1400 + 13 tens + 11; 70000 + 12000 + 1100 + 16 tens + 11; 30000 + 14000 + 1800 + 10 tens + 18; 50000 + 13000 + 1600 + 11 tens + 19; 700000 + 90000 + 12000 + 1100 + 13 tens + 15; 200000 + 160000 + 10000 + 1100 + 18 tens + 16.

VII.—(1) 24917. (2) 17866. (3) 102119. (4) 9496. (5) 2077. (6) 7865. (7) 4961. (8) 17985. (9) 351152. (10) 27166. (11) 172846. (12) 60562. (13) 8309287. (14) 8970598. (15) 3028394. (16) 16548360. (17) 12461. (18) 54015674. (19) 21735. (20) gold 283, silver 568, copper 954: total 1805. (21) 1333, 1720. (22) wheat 4368, barley 10298, oats 3042: total 17708. (23) 3345, 2201, 5858, 4202, 2299. (24) 290 days. (25) £4597. (26) 4314 letters.

VIII.—(1) £2900 19s. 5½d. (2) £11100 15s. 7½d. (3) £3338 3s. 11d. (4) £1243 15s. 2¾d. (5) 16 tons 16 cwt. 2 *qrs. 5 lbs.* (6) 41 lbs. 8 oz. 5 dwt. 17 grs. (7) 4 tons 3 cwt.

1 qr. 26 lbs. 7 oz. 4 drs. (8) 21 miles 6 fur. 33 poles 3 yds. 2 ft. 7 in. (9) 90 acres 2 roods 4 poles 30 yds. (10) 22 qrs. 5 bush. 3 pecks 1 gal. 3 qts. (11) 1 cwt. 3 qrs. 6 lbs. 1 oz. 6 drs. (12) 66 days 17 hours 10 min. (13) 300 acres 0 roods 28 poles. (14) 232 yds. 2 ft. 134 in. (15) 222 gals. (16) 30 hours 43 min. 42 sec. (17) 94 leagues 1 mile 7 fur. 12 poles 1 ft. (18) 382 yds. 1 qr. (19) 10 miles 7 fur. 1 pole 3 ft. 11 in. (20) £1076 4s. 5½d.

X.—(1) 6; 613; 707; 465; 8024; 49724; 4017. (2) 87208; 2563; 8297; 36747; 1789; 3611; 276. (3) 241; 341; 5784; 22706; 3424; 5319. (4) 507; 157; 366; 1083. (5) 37393; 999194; 7055. (6) 37993; 2678; 22815; 1265; 47961; 11644; 37364; 50611; 39410; 7813. (7) 214; 52256; 25480; 12965; 36718; 1773; 497929; 5061. (8) 1959. (9) 71895; 16717. (10) £87332. (11) In 1854. The answers are 1311; 1792; 797; 188; 1669. (12) Greater, 758865; difference, 549183. (13) First, 44; second, 59; third, 15; fourth, 12.

XI.—(1) 4 years 581 days 40 hours 114 minutes; 26 weeks 9 days 28 hours. (2) 4 tons 22 cwt. 5 qrs. 38 lbs.; 46 lbs. 18 oz. 25 dwt. 29 grs. (3) 275 qrs. 10 bush. 5 pecks 3 gals.; 17 hours 61 min. 77 sec.; 10 yds. 4 ft. 20 in. (4) 1 mile 10 fur. 47 poles; 4 miles 1768 yds.

XII.—(1) £51 9s. 0½d.; £108 17s. 3d.; £703 6s. 1d.; £51 9s. 7d.; £702 3s. 9d.; £4808 0s. 7½d.; £220 3s. 9d. (2) £226 15s. 6½d.; £3781 13s. 8¾d.; £2738 3s. 10d.; £2705 3s. 8½d.; £11927 18s. 11¼d.; £10 2s. 7d.; £62671 5s. 3¾d.; £466 12s. 5½d. (3) 21 weeks 3 days 19 hours; 16 miles 3 poles 3 yds. 1 ft. 6 in.; 4 leagues 1 mile 4 fur. 37 yds. 1 ft. 6 in. (4) £1225 3s. 5¼d. (5) 12 cwt. 14 lbs.; 1 qr. 19 lbs. 3 oz.; 4 cwt. 1 qr. 16 lbs.; 5 lbs. 6 oz. 4 drs. (6) 5 miles 4 fur. 23 poles. (7) £513 12s. 11d. (8) 27 tons 13 cwt. 2 qrs. 12 lbs.

XIII.—(1) 9; 61. (2) 31; 3. (3) 182; 707. (4) 391; $x - 6$. (5) $a + 3$; $a - 5$.

XIV.—(1) 18; 3; 79; 40. (2) $b - 3$; $y + 5$; $x - 7$; $n - x$. (3) 9; 9; $m + 6$.

XVI.—(1) 1981; 253145; 11856; 26192; 417760. (2) 187623; 6798; 35209284; 414712368. (3) 4374; 589659; 52776; 6411132; 117148. (4) 3684709; 1006580;

29110844; 5064402. (5) 3394992; 3474: 508374; 19820935; (6) 2482956; 6326883; 4369664; 2188396.

XVII.—(1) 671216; 10152; 197280; 2220288. (2) 3020256; 3059420; 229957; 53374167. (3) 23092488; 1702910; 373095.

XVIII.—(1) 262380; 594800; 24822000. (2) 39792900; 22246000; 26716050000. (3) 1749520000; 15736050000; 1433760000. (4) 23388400; 789920000.

XIX.—(1) 3906; 3525993; 66710074; 7199076; 368244; 62395025; 313632; 141684375. (2) 29462508; 48954719; 51098208; 4003868253; 6535022130. (3) 109157832; 209385353; 461941684; 22218444; 4768389; 19747892. (4) 49615636; 36630; 993720; 11805528; 3601620. (5) 19686992; 355752; 111680278; 22589658; 86208. (6) £256. (7) 7074. (8) 2549470.

XX.—(1) £35 18s. 4d.; £59 2s.; £17 5s. 2¼d. (2) £191 10s. 9d.; £313 19s. 4½d.; £1411 17s. 7½d. (3) £21967 11s. 8d.; £10435 13s. 1½d.; £656181 18s. (4) £534789 4s. 9½d.; £2901990; £3359986 5s. 8¼d. (5) £1602832 0s. 1½d.; £2793993 5s. 3d. (6) 97 acres 1 rood 15 poles; 38 tons 17 cwt. 1 qr. 12 lbs.; 63 miles 5 fur. 24 poles. (7) 485 miles 0 fur. 37 poles 4 yds.; 3 cwt. 1 qr. 15 lbs. 0 oz. 8 dwt. 4 grs.; 270 lbs. 5 oz. 6 drs. 2 scruples. (8) 5 tons 3 cwt. 1 qr. 7 lbs.; 248 tons 15 cwt. 2 qrs. 14 lbs.; 101 tons 14 cwt. 1 qr. 2 lbs. (9) 1 lb. 3 oz. 9 dwt.; 11 lbs. 10 oz. 9 dwt. 12 grs.; 459 lbs. 4 oz. 5 dwt. 22 grs. (10) 125 gals.; 175 gals. 2 qts. (11) 2422 acres 0 roods 12 poles; 705 acres 0 roods 14 poles; 454 acres 0 roods 34 poles. (12) 474 leagues 1 mile 6 fur. 6 poles 1 yd.; 432 miles 7 fur. 12 poles. (13) 87 yds. 0 qrs. 3 nails; 2298 yds. 3 qrs. 2 nails. (14) 213 weeks 5 days 15 hours; 1740 weeks 0 days 18 hours 54 min. (15) 1378 qrs. 5 bush. 2 pecks; 3982 qrs. 7 bush.; 1608 qrs. 3 bush. 3 pecks. (16) £418 19s. 7d.; £867 17s. 8¾d.; £3232 2s. 6d. (17) 9696 acres 34 poles; 10442 acres 12 poles; 20138 acres 1 rood 6 poles. (18) £274 4s. 4½d. (19) £1456 13s. 4d. (20) 598 qrs. 6 bush. 9 qts.

XXI.—(1) 1182600000; 334416720000000; 14916000. (2) 2499000; 386640000; 885530000. (3) 11412162000; 153294000.

XXII.—(1) 474232; 397592; 72744. (2) 724815; 993552;

96831296; 69326. (3) 6714160; 3578126; 56179368; 934974. (4) 77824; 1855365; 874259; 562770. (5) 1102577; 9580977; 16527214011. (6) 12853934; 3994912; 28469426. (7) 7826616; 35466730; 70203285. (8) 91554381; 1196020822; 3705138048. (9) 1737214710; 250771828; 163543296. (10) 6880927680; 258493422; 17136558.

XXIII.—(1) 801504; 533977968; 43785. (2) 710838; 6394903263; 1718188164. (3) 541904185; 7225218. (4) 235246; 5045832; 3752362746.

XXIV.—(1) 7801677; 6975330; 391896702. (2) 4196367; 525408; 680016. (3) 26334396; 5151696; 687160. (4) 26606496; 25536842; 260609202.

XXV.—(1) 2419200; 1468800; 27900; 309600 seconds. (2) 12544 oz.; 7168 lbs.; 392 drs.; 2664 grs. (3) 21120 ft.; 636 in.; 174240 sq. ft.; 9292800 sq. yds. (4) 112 half-pints; 608 pecks; 2016 gills; 5184 pints.

XXVI.—(1) $14592\frac{3}{5}$; 105907; $11306\frac{10}{11}$; $1366\frac{7}{8}$; $49748\frac{1}{11}$: 121639; $54\frac{1}{5}$. (2) $10425\frac{5}{7}$; $7904\frac{1}{11}$; $33535\frac{1}{3}$; 5902; 16583; $8229\frac{1}{4}$. (3) 10972; 120247; $60765\frac{1}{2}$; $6548\frac{1}{5}$; $9121\frac{1}{2}$. (4) $56\frac{2}{5}$; 28; 462; $32\frac{8}{11}$.

XXVII.—(1) $155\frac{22}{28}$; $277\frac{10}{45}$; $11462\frac{34}{84}$. (2) $370\frac{31}{40}$; $3153\frac{52}{120}$; $332\frac{14}{72}$. (3) $26\frac{3}{21}$; $1428\frac{1}{44}$; $201\frac{56}{63}$; $330\frac{24}{35}$. (4) $2201\frac{10}{18}$; $741\frac{6}{82}$; $972\frac{10}{99}$. (5) $1970\frac{2}{42}$; $630\frac{38}{63}$; $1467\frac{12}{56}$. (6) $867\frac{82}{84}$; $827\frac{25}{48}$; $275\frac{14}{108}$.

XXVIII.—(1) $551\frac{337}{807}$; $24987\frac{203}{357}$; $2024\frac{10}{36}$; $8943\frac{31}{57}$; $2212\frac{162}{327}$; $46006\frac{87}{126}$. (2) $52548\frac{21}{78}$; $13833\frac{41}{235}$; $18960\frac{354}{427}$; $965\frac{372}{525}$; $46094\frac{12}{18}$. (3) $1120\frac{48}{65}$; $16047\frac{18}{49}$; $12\frac{3010}{4098}$; $6528\frac{29}{47}$; $170\frac{129}{163}$; $491\frac{38}{87}$; $572\frac{176}{541}$; $694\frac{486}{587}$; $58\frac{571}{678}$. (4) $283\frac{4}{18}$; $23191\frac{4}{29}$; $3726\frac{29}{123}$; $137\frac{22}{372}$; $529\frac{183}{526}$; $647\frac{57}{324}$. (5) $963\frac{11}{31}$; $18813\frac{17}{27}$; $7\frac{2}{51}$; $1389\frac{38}{45}$. (6) $982\frac{4}{17}$; $63\frac{240}{263}$; $147\frac{60}{81}$. (7) $17\frac{23}{44}$; $69\frac{8}{21}$.

XXIX.—(1) £180 17s. 11d.; £8 15s. $0\frac{3}{4}$d.; £97 16s. $10\frac{3}{4}$d. (2) £59 11s. 3d.; £45 15s. $7\frac{1}{4} + \frac{1}{3}$; £142 17s. $1\frac{1}{2}$d. $+ \frac{6}{7}$. (3) £896 7s. $11\frac{1}{2}$d.; £784 6s. $11\frac{1}{2}$d. $+ \frac{1}{4}$; £697 3s. $11\frac{1}{2}$d. $+ \frac{4}{9}$. (4) £10 12s. $7\frac{3}{4}$d. $+ \frac{9}{28}$; £10 7s. $7\frac{1}{2}$d. $+ \frac{33}{38}$. (5) £2 9s. 5d. $+ \frac{41}{514}$; £91 15s. $5\frac{1}{2}$d. $+ \frac{10}{17}$; £5 7s. $5\frac{1}{4}$d. $+ \frac{479}{510}$. (6) £213 1s. $2\frac{3}{4}$d.; £23 4s. $11\frac{3}{4}$d. $+ \frac{67}{137}$; £44 12s. $6\frac{1}{2}$d. $+ \frac{34}{49}$. (7) £47 5s. 10d. $\frac{158}{631}$; £955 10s. 5d. $+ \frac{8}{65}$; £2 0s. $0\frac{3}{4}$d. $+ \frac{5}{9}$. (8) 4 cwt. 3 qrs. 16 lbs. 10 oz. $10\frac{2}{3}$ drs.; 14 cwt. 2 qrs. 20 lbs. 4 oz. $6\frac{18}{29}$ drs.; 1 cwt. 1 qr. 21 lbs. 12 oz. $12\frac{4}{5}$ drs. (9) 9 oz. 8 dwt.; 1 lb. 1 oz. 12 dwt.

$21\frac{1}{11}$ grs.; 2 oz. 15 dwt. $17\frac{1}{125}$ grs. (10) 1 mile 0 fur. 26 poles 0 yds. 1 ft. 0 in. $1\frac{1}{8}$ barleycorns; 8 miles 3 fur. 1 pole 4 yds. 0 ft. 2 in. $1\frac{1}{15}$ barleycorns; 21 poles 1 yd. 2 ft. $4\frac{2}{7}$ in. (11) 4 acres 1 rood; 2 acres 3 roods 13 poles 10 yds. 2 ft. 36 in.; 1 acre 2 roods 7 poles 8 yds. 2 ft. 36 in.; 1 acre 1 rood 9 poles 6 yds. 8 ft. $119\frac{1}{15}$ in. (12) 46 qrs. 4 bush. 0 pecks $1\frac{1}{4}$ gal.; 5 bush. $1\frac{5}{27}$ pecks; 1 qr. 5 bush. 1 peck $1\frac{1}{3}$ gal.

XXX.—(1) 1100; 7661[illegible]; 2790[illegible]. (2) 4893[illegible]; 1000; 18800[illegible]. (3) 470[illegible]; 8274[illegible]; 727[illegible]. (4) $153\frac{6}{40}$; 115[illegible]; 8168[illegible]. (5) 1057[illegible]; 6270; 127. (6) 2830[illegible]; 59[illegible]. (7) 82[illegible]; 888[illegible]; 120[illegible]. (8) 224; 347[illegible]; 19033[illegible]. (9) 2181[illegible]. (10) 1732[illegible] fourpenny-pieces, 1155 sixpences. (11) 7556[illegible]. (12) 8310[illegible] half-guineas; £912[illegible]. (13) $107\frac{1}{11}$. (14) 178[illegible]. (15) 7040.

XXXVI.—(1) £1 5s. 8[illegible]d.; £858 8s. $5\frac{1}{2}$d. (2) £29 2s. $6\frac{3}{4}$d.; £58 5s. 1[illegible]d.; £116 10s. 8d. (3) 6 hours 37 min. 36 sec.; 110 days 9 hours 20 min.; 1145 years 18 weeks 4 days. (4) 2432 crowns 10[illegible]d. (5) 81 miles 2 fur. 1 pole 8 yds. $2\frac{1}{2}$ ft.; 23 yds. 2 ft. 11 in. (6) 12 tons 9 cwt. 8 qrs. 17 lbs. 2 oz. (7) 1092. (8) 1[illegible] troy, 1[illegible] avoirdupois.

XXXVII.—(1) 8 × 8; 16 × 4; 8 × 7. (2) [illegible]; 8 × 4; [illegible] × [illegible]. (3) [illegible]; [illegible]; [illegible].

XXXVIII.—[illegible] (2) [illegible]; 8; 63. (3) 1; [illegible] (4) [illegible]

[illegible]

XLVI.—(1) 8^8; 12^{17}; 15^6. (2) a^{13}; p^6; 9^{m+n+p}. (3) 6^9; 15^7; $\frac{3^2 \times 7}{5}$. (4) $\frac{20^3 \times 25}{2}$; $\frac{5^{12}}{9}$; $16^{m-3} \times 14^{n-1}$. (5) a^3; $a^{m-1} b^{m-2}$; $p^{m-n} 9^{m-4}$.

XLVII.—(1) 140535. (2) $56\frac{8}{9}$; $41\frac{87}{85}$; $1327\frac{1}{5}$. (3) 6s. $5\frac{1}{2}$d. + $\frac{1}{3}$. (4) £25 1s. 8d.; £32 16s. 3d. (5) £17 4s. 9d. (6) 3s. $3\frac{1}{4}$d. $\frac{11}{23}$; £2 18s. $9\frac{3}{4}$d. $\frac{9}{17}$. (7) 622640. (8) 448416. (9) 80; 940; 175360; 2577755; 2668; 257041; $204\frac{5}{26}$. (10) $10195\frac{75}{87}$. (11) 37128. (12) 525948. (13) 1407. (14) 4528. (15) Belgium 320 francs 67 centimes; Prussia 85 thalers; Frankfort 152 florins 42 kreutzers. (16) 118839. (17) 96. (18) 976. (19) $380\frac{4}{17}$. (20) 1 acre 3 roods 21 poles 4 yds. 4 ft. 48 in. (21) 11 lbs. 9 oz. 19 dwt. 12 grs. (22) £819 12s. $8\frac{1}{4}$d. (23) 2 oz. 7 dwt. $10\frac{2}{13}$ grs. (24) 5 lbs. 6 dwt. $22\frac{1}{2}$ grs. (25) 9 miles 1 fur. 25 poles 2 yds. 1 ft. 6 in. (26) 139920. (27) £29 4s. $4\frac{1}{2}$d. (28) 3 ft. $6\frac{4}{5}$ in. (29) 25637 + $\frac{3}{4}$d. (30) £1726 13s. 4d. (31) 1s. $1\frac{1}{2}$d. (32) £281 5s. 0d. (33) 1st = £5333 6s. 8d.; 2nd = £8000 0s. 0d.; 3rd = £10666 13s. 4d. (34) 537. (35) 4 miles 0 fur. 31 poles 4 yds. 1 ft. 7 in. $\frac{29}{31}$. (36) £1 5s. $9\frac{3}{73}$d. (37) 49116. (38) 623 oz. 18 dwt. 18 grs. (39) $522\frac{199}{225}$. (40) 1353 yds. 2 qrs. 2 nails. (41) £5829 4s. 7d. (42) 276 days. (43) 528093440. (44) 20475. (45) £66 7s. 5d. (46) £14595. (47) £6 17s. $7\frac{1}{2}$d. (48) $625\frac{45}{99}$. (49) $313\frac{1}{3}$ gals. (50) $34\frac{2}{7}$. (51) $179\frac{105}{132}$. (52) £0 0s. 8d. $\frac{384}{420}$. (53) £2 3s. $4\frac{1}{2}$d. $\frac{18}{63}$. (54) 13.

XLVIII.—(1) 5; 36; 30. (2) 7; 11; 5. (3) 8; 9; 12. (4) 5; 4; 1. (5) 1; 1; 4. (6) 2; 17; 1.

XLIX.—(1) 25; 8; 8. (2) 23; 2; 2. (3) 7; 12; 3.

L.—(1) 347; 2 × 29; 7 × 7 × 2 × 2. (2) 13 × 79; 2 × 41 × 43; 2 × 3 × 683. (3) 7 × 13 × 79; 31 × 211; 4127. (4) 3 × 7723; 3 × 3 × 3 × 7 × 29; 2 × 35543.

LII.—(1) 175; 216; 693. (2) 22734; 2100; 300. (3) 3936; 3248088; 1488. (4) 9165; 496; 150.

LIII.—(1) 150; 924; 2320. (2) 5040; 270. (3) 39270; 9500652. (4) 2520; 360. (5) 4149360; 14364. (6) 25256351428; 441000.

LIV.—(1) $\frac{89}{12}$; $\frac{76}{9}$; $\frac{40}{11}$. (2) $\frac{221}{7}$; $\frac{817}{15}$; $\frac{451}{19}$. (3) $\frac{1111}{9}$; $\frac{276}{13}$; $\frac{13}{7}$. (4) $\frac{377}{8}$; $\frac{42}{19}$; $\frac{4941}{27}$. (5) $\frac{150}{7}$; $\frac{130}{9}$; $\frac{mn + a}{n}$.

LV.—(1) $4\frac{5}{9}$; $7\frac{3}{5}$; $18\frac{1}{13}$. (2) $6\frac{1}{3}$; $37\frac{5}{11}$; $25\frac{54}{125}$. (3) $118\frac{3}{4}$.

$12\frac{1}{6}$; $124\frac{4}{5}$. (4) $123\frac{14}{17}$; $70\frac{59}{105}$; $22\frac{55}{167}$. (5) $15\frac{253}{263}$; $17\frac{310}{341}$; $227\frac{7}{15}$. (6) $90\frac{2}{27}$; $4\frac{4}{25}$; $53\frac{23}{41}$.

LVI.—a. (1) $\frac{19}{7}$; $\frac{2}{3}$; $\frac{1}{3}$. (2) $\frac{24}{7}$; $\frac{21}{19}$; 4. (3) $\frac{28}{19}$; 3; $\frac{47}{25}$. (4) $\frac{16}{6}$; 4; $\frac{221}{54}$. (5) $\frac{14}{6}$; $\frac{9}{2}$; $\frac{14}{3}$. b. (1) $\frac{7}{90}$; $\frac{3}{70}$; $\frac{2}{17}$. (2) $\frac{7}{36}$; $\frac{3}{25}$; $\frac{41}{640}$. (3) $\frac{3}{19}$; $\frac{2}{47}$; $\frac{47}{370}$. (4) $\frac{1}{23}$; $\frac{2}{75}$; $\frac{16}{133}$. (5) $\frac{1}{75}$; $\frac{4}{57}$; $\frac{2}{41}$. (6) $\frac{2}{147}$; $\frac{23}{1000}$; $\frac{47}{590}$.

LVII.—b. (1) $\frac{51}{3}$; $\frac{136}{8}$; $\frac{119}{7}$. (2) $\frac{20}{5}$; $\frac{36}{9}$; $\frac{24}{6}$. (3) $\frac{96}{8}$; $\frac{36}{3}$; $\frac{48}{4}$. (4) $\frac{60}{12}$; $\frac{48}{12}$; $\frac{36}{9}$. (5) $\frac{189}{21}$; $\frac{27}{3}$; $\frac{126}{14}$.

LVIII.—(1) $\frac{35}{63}$; $\frac{28}{753}$; $\frac{2}{4}$. (2) $\frac{57}{209}$; $\frac{4}{9}$; $\frac{100}{793}$. (3) $\frac{205}{274}$; $\frac{17}{225}$; $\frac{112}{113}$. (4) $\frac{55}{114}$; $\frac{74}{85}$; $\frac{91}{96}$. (5) $\frac{70}{133}$; $\frac{3}{80}$; $\frac{7}{13}$. (6) $\frac{50}{1512}$; $\frac{227}{3465}$; $\frac{44}{63}$.

LIX.—(1) $\frac{90}{280}$; $\frac{105}{280}$; $\frac{224}{280}$. (2) $\frac{1470}{2920}$; $\frac{504}{2920}$; $\frac{180}{2920}$. (3) $\frac{1700}{6460}$; $\frac{2280}{6460}$; $\frac{969}{6460}$. (4) $\frac{8704}{15232}$; $\frac{14280}{15232}$; $\frac{5712}{15232}$; $\frac{3584}{15232}$. (5) $\frac{4368}{7488}$; $\frac{1728}{7488}$; $\frac{2340}{7488}$; $\frac{4992}{7488}$. (6) $\frac{102600}{376200}$; $\frac{12540}{376200}$; $\frac{59400}{376200}$; $\frac{175500}{376200}$. (7) $\frac{504}{3780}$; $\frac{810}{3780}$; $\frac{1680}{3780}$. (8) $\frac{1615}{5168}$; $\frac{1088}{5168}$; $\frac{912}{5168}$. (9) $\frac{5852}{7524}$; $\frac{6732}{7524}$; $\frac{7182}{7524}$. (10) $\frac{518700}{678300}$; $\frac{357800}{678300}$; $\frac{209349}{678300}$; $\frac{214200}{678300}$. (11) $\frac{31360}{67200}$; $\frac{26880}{67200}$; $\frac{12600}{67200}$; $\frac{24000}{67200}$. (12) $\frac{54000}{306000}$; $\frac{102000}{306000}$; $\frac{265200}{306000}$; $\frac{24480}{306000}$.

LX.—(1) $\frac{6}{30}$; $\frac{12}{30}$. (2) $\frac{21}{36}$; $\frac{8}{36}$. (3) $\frac{126}{420}$; $\frac{84}{420}$; $\frac{240}{420}$. (4) $\frac{140}{252}$; $\frac{36}{252}$; $\frac{147}{252}$. (5) $\frac{80}{360}$; $\frac{90}{360}$: $\frac{288}{360}$. (6) $\frac{20}{120}$; $\frac{36}{120}$; $\frac{30}{120}$. (7) $\frac{70}{525}$; $\frac{21}{525}$; $\frac{475}{525}$. (8) $\frac{32}{176}$; $\frac{33}{176}$; $\frac{56}{176}$. (9) $\frac{288}{1008}$; $\frac{168}{1008}$; $\frac{882}{1008}$; $\frac{432}{1008}$. (10) $\frac{20}{120}$; $\frac{70}{120}$; $\frac{56}{120}$; $\frac{18}{120}$. (11) $\frac{108}{1800}$; $\frac{300}{1800}$; $\frac{700}{1800}$; $\frac{225}{1800}$. (12) $\frac{900}{2100}$; $\frac{600}{2100}$; $\frac{18}{2100}$.

LXI.—(1) $\frac{34}{35}$. (2) $\frac{167}{204}$. (3) $1\frac{143}{300}$. (4) $6\frac{23}{60}$. (5) $23\frac{88}{117}$. (6) $1\frac{599}{1540}$. (7) $10\frac{243}{1015}$. (8) $7\frac{9}{20}$. (9) $6\frac{49}{60}$. (10) $1\frac{1539}{10000}$. (11) $\frac{2523}{2808}$. (12) $5\frac{17}{60}$. (13) $99\frac{95}{84}$. (14) $17\frac{389}{552}$. (15) $102\frac{8}{663}$. (16) $13\frac{17}{20}$. (17) $22\frac{1023}{1053}$. (18) $17\frac{37}{60}$. (19) The whole.

LXII.—(1) $\frac{13}{60}$. (2) $\frac{7}{12}$. (3) $\frac{24}{133}$. (4) $\frac{5}{12}$. (5) $\frac{1}{6}$. (6) $1\frac{199}{228}$. (7) $3\frac{37}{46}$. (8) $\frac{19}{126}$. (9) $6\frac{11}{20}$. (10) $10\frac{29}{35}$. (11) $7\frac{31}{35}$. (12) $8\frac{577}{1236}$. (13) $147\frac{1}{8}$. (14) $14\frac{1}{45}$. (15) $5\frac{73}{112}$. (16) Sum $1\frac{1}{3}$, diff. $\frac{1}{6}$. (17) $16\frac{59}{60}$. (18) $72\frac{5087}{5544}$.

LXIII.—(1) $\frac{8}{35}$; $\frac{12}{85}$; $3\frac{4}{7}$. (2) $\frac{12}{25}$; $6\frac{1}{5}$; $18\frac{23}{45}$. (3) $\frac{15}{616}$; $\frac{5}{72}$; $\frac{2}{7}$. (4) $\frac{9}{28}$; $7\frac{1}{2}$; $7\frac{99}{105}$. (5) $\frac{212}{945}$; $\frac{1367}{29070}$. (6) $6\frac{13}{42}$; $\frac{229}{5355}$. (7) $25466\frac{6}{35}$; $6526\frac{127}{168}$; $21\frac{385}{391}$. (8) $21\frac{3}{7}$; $5\frac{5}{8}$; $\frac{63}{200}$. (9) $\frac{1036}{27225}$. (10) $63\frac{86}{153}$. (11) $1\frac{1881}{14560}$.

LXIV.—(1) $\frac{1}{7}$; 9; $\frac{15}{14}$. (2) $\frac{5}{13}$; $\frac{6}{17}$; $\frac{11}{160}$. (3) $\frac{9}{190}$; $\frac{7}{8}$; $\frac{6}{649}$.

LXV.—(1) $\frac{18}{35}$; 2; $1\frac{11}{27}$. (2) $1\frac{19}{50}$; $43\frac{21}{25}$; $6\frac{33}{35}$. (3) $\frac{9}{14}$; $2\frac{6}{17}$; 5; $83\frac{359}{380}$. (4) $4\frac{197}{208}$; $74\frac{1}{4}$; $18\frac{381}{413}$. (5) $1\frac{173}{252}$; $106\frac{53}{819}$.

(6) $\frac{12}{59}$. (7) $2\frac{38}{49}$. (8) $24\frac{7}{13}$. (9) $\frac{238}{675}$. (10) $\frac{20723}{51480}$. (11) $\frac{5784}{6391}$. (12) $8\frac{177}{1376}$.

LXVI.—(1) $\frac{4}{2565}$ and $\frac{1}{49}$. (2) $\frac{5}{216}$; $\frac{5}{72}$; $22\frac{2}{3}$; $122\frac{2}{9}$; $366\frac{2}{3}$. (3) $\frac{4}{70}$ of £1; $\frac{4}{35}$ of a half-sovereign; $\frac{8}{35}$ of 5s.; $\frac{16}{35}$ of 2s. 6d.; $\frac{8}{7}$ of 1s.; $\frac{16}{7}$ of 6d.; $\frac{24}{7}$ of 4d.; $\frac{32}{7}$ of 3d.; $\frac{96}{7}$ of 1d.; $\frac{192}{7}$ of $\frac{1}{2}$; $\frac{384}{7}$ of $\frac{1}{4}$. (4) $\frac{1}{2304}$. (5) $\frac{1}{2176}$ of cwt.; $\frac{7}{136}$ of lb. (6) $\frac{5}{2772}$. (7) $\frac{36200}{73}$. (8) $\frac{82}{105}$ of 15s.; $\frac{41}{70}$ of £1; $\frac{41}{350}$ of £5. (9) $\frac{480}{13}$ of a minute; $\frac{1}{39}$ of a day; $\frac{1}{273}$ of a week. (10) $\frac{144}{25}$ of a pint; $\frac{3}{350}$ of a puncheon; $\frac{1}{175}$ of a pipe. (11) $\frac{9}{1120}$. (12) $\frac{20}{326840}$. (13) $\frac{19}{32}$ of an acre; $\frac{19}{20480}$ of a mile. (14) $\frac{1}{7500}$.

LXVII.—(1) $\frac{5}{2}$, $\frac{7}{3}$, $\frac{12}{5}$, $\frac{43}{15}$; $\frac{1}{3}$, $\frac{9}{28}$, $\frac{10}{31}$. (2) $\frac{5}{4}$, $\frac{56}{45}$, $\frac{117}{94}$, $\frac{290}{233}$, $\frac{407}{327}$; $\frac{6}{7}$, $\frac{13}{5}$, $\frac{32}{37}$. (3) $\frac{4}{5}$; $\frac{9}{2}$, $\frac{13}{3}$, $\frac{35}{8}$, $\frac{83}{19}$. (4) $\frac{1}{6}$, $\frac{1}{7}$, $\frac{4}{27}$, $\frac{21}{142}$, $\frac{46}{311}$; $\frac{15}{4}$, $\frac{49}{15}$. (5) $\frac{1}{22}$, $\frac{2}{45}$, $\frac{7}{157}$, $\frac{93}{2086}$, $\frac{193}{4329}$; $\frac{1}{199}$, $\frac{83}{16518}$, $\frac{84}{16717}$. (6) $\frac{1}{2}$, $\frac{42}{83}$; $\frac{6}{1}$, $\frac{17}{3}$, $\frac{23}{4}$, $\frac{224}{39}$, $\frac{247}{43}$. (7) $\frac{14}{13}$, $\frac{29}{27}$, $\frac{72}{67}$, $\frac{173}{161}$, $\frac{418}{389}$; $\frac{6}{5}$, $\frac{7}{6}$, $\frac{55}{47}$, $\frac{62}{53}$. (8) $\frac{2}{3}$, $\frac{11}{16}$.

LXVIII.—(1) 2s. $3\frac{9}{16}$d., and £3 10s. 0d. (2) 2 qrs. 24 lbs., and 5 cwt. 1 qr. $9\frac{1}{3}$ lbs. (3) 56, and $2\frac{19}{29}$ miles. (4) £33 16s. $3\frac{15}{19}$d., and £4 19s. $8\frac{12}{23}$d. (5) £3 13 $3\frac{3}{8}$d., and £10 13s. $10\frac{3}{8}$d. (6) £68 8s. $8\frac{8}{19}$d., and £12 18s. $10\frac{3}{7}$d. (7) £9 16s. 0d., and £27 17s. $9\frac{43}{49}$d. (8) 5 hhds. 1 kil. 6 gals., and 17 hhds. 1 runlet $7\frac{1}{5}$ gals. (9) 1 mile 1 fur. 5 poles 3 yds. 2 ft. $9\frac{3}{7}$ in., and 3 roods 18 poles 20 yds. 1 ft. 72 in., and 373 acres 1 rood 13 poles 10 yds. 0 ft. 108 in. (10) 5 bush. 3 pecks 1 gal., and 1 bush. 3 pecks $1\frac{1}{17}$ gals., and 1 bush. 3 pecks 0 gals. 2 quarts $1\frac{5}{7}$ pints. (11) £1 4s. $0\frac{96}{133}$d., and £1 13s. $2\frac{26}{53}$d., and $3\frac{17}{21}$d. (12) 2 weeks 6 days, and 8 weeks $\frac{2}{13}$ of a day, or $56\frac{2}{13}$ days. (13) 3 miles 0 fur. 27 poles 2 yds. $1\frac{1}{14}$ ft., and 5 fur. 25 poles 4 yds. $2\frac{19}{34}$ ft. (14) 5 cwt. 2 qrs. $16\frac{8}{17}$, and 2 qrs. $1\frac{1}{55}$ lbs. (15) 18s. $5\frac{613}{2431}$d.

LXIX.—(1) 6 hours, and 3 days 12 hours. (2) 4s. 3d., and £13 14s. 6d. (3) 45 yds., and $33\frac{1}{3}$ ft. (4) 1 lb. 0 oz. 3 dwt. 16 grs. (5) 21 lbs. 6 oz. 8 drs., and 6 cwt. 2 qr. 24 lbs. 14 oz. $3\frac{5}{9}$ drs. (6) $26\frac{1}{15}$ and $7\frac{9}{125}$. (7) £15. (8) 6 acres 1 rood 20 perches.

LXX.—(1) $\frac{13}{648}$. (2) $\frac{161}{1008}$. (3) $\frac{77}{3384}$. (4) $\frac{409}{11200}$. (5) $\frac{1}{360}$. (6) $\frac{2063}{24000}$; $\frac{131}{6000}$. (7) $\frac{17}{288}$. (8) $\frac{161}{240}$; $\frac{23}{144}$. (9) $\frac{13}{21136}$. (10) $\frac{1157309}{5989200000}$.

LXXI.—(1) £53892 2s. 6d.; £5600 9s. $9\frac{3}{4}$d. (2) £70331 14s. 8d.; £10854 15s. (3) £130714 4s. $1\frac{1}{2}$d.; £8676 5s. (4) £33024 19s. 7d.; £115372 3s. $10\frac{1}{2}$d. (5) £295542 7s. $3\frac{3}{4}$d.; £273004 15s. $10\frac{1}{2}$d. (6) £554 18s. 6d.; £16066 10

(7) £4384 15s. 10¾d., and £63722 10s. 0½d. (8) £604 19s. 6¾d., and £10833 10s. 7½d. (9) £6259 7s. 0½d., and £306390 3s. 0d. (10) £13267 0s. 11¼d., and £40690 1s. 8d. (11) £26995 3s. 0d., and £16703 18s. 10½d. (12) £2206955 19s. 10¼d., and £35451 18s. 2d. (13) £9343 0s. 0¾d., and £1336 3s. 3½d. (14) £348046 2s. 10d., and £76173 3s. 6d. (15) £55234 1s. 3d., and £1946 11s. 9d. (16) £281 18s. 4¼d., and £29958 13s. 1¾d. (17) £327951 6s. 10d., and £17705 3s. 8¼d. (18) £31993 2s. 6¾d., and £2760 3s. 3d. (19) £578708 13s. 4d., and £217975 15s. 4d. (20) £462212 4s. 3d., and £1168918 16s. 9¼d.

LXXII.—(1) £5 3s. 7½d. (2) £48 0s. 7½. (3) £532 5s. 5d. (4) £9 1s. 3¾d. (5) £31 3s. 6¾d. (6) £207 10s. 6½d. (7) £153 10s. 4½d. (8) £461 7s. 10½d. (9) 16s. 9½d. (10) 5s. 6¼d. (11) 5d. (12) 12s. 8¼d. (13) £46 5s. 2¾d. (14) £339 1s. 3¾d. (15) £145 16s. 5d. (16) £165 16s. 6¼d.

General Exercises on Vulgar Fractions.

(1) £208 14s. $10\frac{5}{11}$d. (2) 37½ days. (3) $13\frac{941}{8000}$. (4) $\frac{1029}{160}$. (5) $\frac{2413}{320}$. (6) $11\frac{5}{8}$. (7) $2\frac{13}{40}$; $\frac{193}{1235}$. (8) $1\frac{22}{105}$; $7\frac{14}{15}$. (9) $\frac{307}{496}$. (10) $\frac{1}{1} + \frac{1}{17} + \frac{1}{4}$. (11) £16 11s. $8\frac{11}{12}$d. (12) £7 16s. $5\frac{39}{80}$d. (13) $\frac{4}{13}$. (14) $2\frac{425}{748}$; $46\frac{25}{41}$. (15) $\frac{4623}{10472}$. (16) $1913\frac{137}{280}$. (17) $\frac{1}{6}$. (18) $2\frac{97}{144}$. (19) 112. (20) 1100. (21) 42. (22) $3\frac{127}{165}$; $16\frac{349}{532}$. (23) £143 10s. 4d. (24) £311 6s. $5\frac{49}{67}$d. (25) $\frac{945}{1804}$. (26) $7\frac{27}{83}$. (27) $7\frac{27}{83}$. (28) £1 10s. 4d. (29) $\frac{13}{207}$. (30) $72\frac{1901}{2464}$. (31) $\frac{638}{1102}$; $\frac{684}{1102}$; $\frac{667}{1102}$. (32) 1s. $0\frac{12}{19}$d.; 1s. $0\frac{3}{5}$d.; 1s. $1\frac{1}{7}$d. (33) $1\frac{61}{64}$. (34) £884 18s. 0¼d. (35) £$21\frac{27}{49}$. (36) £180. (37) $\frac{559}{90000}$. (38) 27 acres 16 roods.

LXXVII.—(1) ·57142+; 1·82746; 1·06896. (2) ·66936; ·8144; ·88888. (3) ·58333; ·72; 2. (4) ·92187; ·168; ·88888. (5) 1·0625; ·135; ·15813. (6) ·16; ·8; ·36. (7) 2·875; 13·4; 122·77777+. (8) ·52941+; ·42490+; ·102+. (9) ·3125; 12·75; 2·57142+. (10) ·17857+; ·75789+. (11) ·16806; ·86806+. (12) ·41379+; 26.

LXXVIII.—(1) $\cdot\dot{6}\dot{3}6363$; $\cdot\dot{5}5555$; $1\cdot\dot{0}952\dot{3}809$. (2) $\cdot\dot{5}7142\dot{8}$; $\cdot58\dot{3}$; $\cdot\dot{4}2105263157894736\dot{8}$. (3) $\cdot\dot{3}8461\dot{5}$; $\cdot\dot{3}52941176470588\dot{2}$; $\cdot2\dot{6}$. (4) $\cdot\dot{7}39130434782608695652\dot{1}$; $\cdot\dot{6}3157894736842105\dot{2}$; $\cdot\dot{2}\dot{2}$. (5) $\cdot\dot{4}70588235294117\dot{6}$; $\cdot\dot{3}8461\dot{5}$; $\cdot\dot{4}2857\dot{1}$. (6) $\cdot\dot{6}\dot{6}$; $\cdot\dot{6}\dot{6}$; $\cdot\dot{8}\dot{8}$.

LXXIX.—(1) $\frac{47}{99}$; $\frac{5}{9}$; $\frac{5}{90}$. (2) $\frac{327}{999}$; $\frac{95238}{999999}$; $\frac{714285}{999999}$. (3) $\frac{185}{99900}$; $\frac{3129}{9990}$; $1\frac{6946}{9900}$. (4) $\frac{28144}{99900}$; $3\frac{411}{990}$; $\frac{70188}{99900}$. (5) $6\frac{38}{999}$; $\frac{7054}{9900}$; $\frac{80483}{99900}$. (6) $\frac{13574}{99900}$; $\frac{719}{990}$; $6\frac{27}{990}$. (7) $1\frac{27289}{99900}$; $45\frac{458}{900}$; $\frac{827}{999}$. (8) $\frac{7025}{9990}$; $6\frac{37}{90}$; $7\frac{43}{990}$. (9) $27\frac{56}{990}$; $3\frac{75}{9900}$; $4\frac{127}{999}$. (10) $11\frac{372}{999}$; $4\frac{152}{900}$; $\frac{50972}{99990}$. (11) $\frac{7}{990}$.

LXXXII.—(1) 587·167. (2) 303·4867. (3) 2064·1614. (4) 150·9036. (5) 1231·2914. (6) 235·1545. (7) 915·2858. (8) 1370·876842. (9) 1045·9222. (10) 532·0767. (11) 1265·6408. (12) 12890·28028. (13) ·217047. (14) 24·260934. (15) 5·065874.

LXXXIII.—(1) 708·7786; 215·557. (2) 52·3416; 80·263. (3) 2090·35238; 17·82. (4) 149·193; 216·06528. (5) 1158·002; 38·835. (6) ·999; 729·7913. (7) 2123·4922; 62·33155. (8) 2141·181; 41·48357. (9) 871·2199; 277·01714. (10) 28·0862; 319·6. (11) ·71373. (12) 68·9$\dot{6}$. (13) 4·26$\dot{2}$0$\dot{2}$; 72·5847. (14) 27·912967. (15) ·6334. (16) 33116. (17) ·95. (18) ·616174285. (19) Former by ·03173. (20) 7·717.

LXXXIV.—(1) 1776·73; 6531·71. (2) 297·0225; 254·2512. (3) 1221815·868; 836·5427. (4) 8·9332828; ·1360485. (5) 19533·176; 424097·46822. (6) 117467·0923216; 150352·146945. (7) 98·2602; 77·8321752. (8) 379·1172; 17·303. (9) 4729·78826304; ·2628. (10) 143·507702; 711·59055. (11) 886761·65808; 296·4114. (12) 475·135024. (14) £813·$\dot{2}2\dot{2}$. (15) 4090·714. (16) £18003·382575. (17) £236·521728.

LXXXV.—(1) 6·263; ·24094; 15·097202. (2) 3·0495; ·31008; ·15421. (3) ·83235; ·603738; 4·98572. (4) 78·1482; 43·0547; ·0246. (5) ·0049876; 112·783; ·01767. (6) ·000139; 1·7453; ·2885428. (7) £53·508571. (8) Each man had ·96375, and paid £1·1515625.

LXXXVI.—(1) 5316·883+; 25·$\dot{7}\dot{5}$; 5·8. (2) 2006·99+; 1000; 5·3623188. (3) 111·81102+; 1·356633+; 176·591375+. (4) 129·903703+; 3448·$\dot{1}4\dot{8}$; 69523·33664. (5) 20432·8574; 10; 100. (6) 107·3379; 13·031114. (7) 7·50084; 75·043717. (8) 23·20518; ·20458. (9) 2073·34456. (10) 8643·402399.

LXXXVII.—(1) ·01, and 17·617, &c., and 11·8235294. (2) 47·072, &c., and ·01722583, and ·2195956. (3) 33·0457339449, and

256·5920833, &c., and 36·34, &c. (4) 2·405283019, and 6142·3, and 6·969923982. (5) 19·5383, &c., and ·6393226, and ·147245413. (6) ·968965517, and 2·93685039, and 6·48354838. (7) 405·7604395, and 29·31328671. (8) ·000932808, &c., and 18·79425004. (9) 7·9161181, and 17·6031584. (10) ·019883481, and ·0448478229, and 5·1249429.

LXXXIX.—(1) 10·019, and 1198·784, and 3·650. (2) 341·071, and 788·424, and 8·929. (3) 572·931, and 1366·360, and 199·248. (4) 22·142, and 43·897, and 1·756. (5) 78·832, and 13·841. (6) 333·795, and 292·045. (7) 132·351, and 2·79.

XC.—(1) 1·972, and 3·678, and 1·82. (2) ·034, and 3·270. (3) 2·383, and ·033, and 6·402.

XCI.—(1) ·00925 fur.; 0015625 of mile; ·000385416 of league; 2·035 of yd.; 6·105 of ft.; 73·26. (2) ·0375£; ·075 of 10s.; ·15 of crown; ·3 of half-crown; ·375 of florin; ·75 of 1s.; 1·5 of 6d.; 2·25 of groat; 3·0 of 3d.; 9·0 of 1d.; 18·0 of halfpenny; 36·0 of farthing. (3) 97·44; ·00005138 and ·005138 of a day. (4) ·023 of a qr.; ·736 of a peck; 11·776 of a pint. (5) 3·9 of 3d.; ·39 of 2s. 6d., and ·065 of 15s. (6) ·074865 of a £10 note, and 718·704 of a farthing. (7) ·1675 of £1; ·335 of 10s., and 80·4 of halfpenny. (8) 3·5 of a florin. (9) ·000241071 of 11 cwt. (10) ·006203125 bush. (11) 609·6 gr.; 30·48 scr., and 25·4 dwt. (12) ·0001523437 of a square mile.

XCII.—(1) 17 weeks 6 days 9 hours 33 min. 7·2 sec.; 1 pint 1·696 gills; 32 weeks 5 days 5 hours 19 min. 40·8 sec. (2) £2 3s. 6¼d.; £5 14s. 6½d.; 16s. 5¼d. + ¾. (3) £17 10s. 0½d.; £1 12s. 6¼d.; £4 15s. 7d. (4) 13s. 7d.; £2 9s. 4¾d.; £1 5s. 7d. (5) £103 9s. 7d.; £2 3s. 8¼d.; £7 1s. 11d. (6) £7 16s. 10d.; £303 15s. 11d.; £2 1s. 8¾d. (7) 29 cwt. 1 qr. 8 lbs. 4 oz. 9 drs.; 5 cwt. 3 qrs. 13 lbs. 7 oz.; 2 qrs. 27 lbs. 2 oz. 8 drs. (8) 3s. 6d; 19s. 5¼d.; 2s. 2d. (9) 3 days 6 hours 28 min. 48 sec.; 4 weeks 5 days 2 hours 18 min. 14 sec.; 39 hours 16 min. 40 sec. (10) 1 mile 2 fur. 12 poles 4 yds. 1 ft. 2½ in.; 27 fur. 11 poles 1 yd. 3 in.; 3 leagues 1 fur. 8 poles. (11) 27 yds. 1 ft. 1 in.; 8 lines 4 poles 1 yd. 2·8 in.; 282 miles 5 fur. 30 poles 2 yds. *7 in.* (12) 37 acres 1 rood; 48 acres 3 roods 35 poles 15 yds. 6 ft.; *82 miles* 25 roods 10 yds. 2 ft. 81 in. (13) 13 yds. 3 ft. 60 in.;

14 miles 150 acres 3 roods 16 poles 21 yds. 2 ft. 95 in.; 3 roods 36 poles 14 yds. 4 ft. 97 in. (14) 17 gals. 3 qts. 0 pts. 2 gills; 212 gals. 1 qt. 0 pts. 1 gill; 1 gal. 0 pts. ·56 gills. (15) 29 qrs. 5 bush. 3 pecks 3 qts. 2 pecks 1 gal.; 53 qrs. 1 bush. 2 pecks 2 qts. (16) 23 yds. 1 qr. 2 nails; 1 yd. 3 qrs. 2 in.; 8 yds. 1 qr. 1 in.

XCIII.—(1) 17·540625; 2·28125; 8·7677. (2) 25·93125; 1·52395; 297·53125. (3) 17·767856; ·83035; 3·8169. (4) 1179·6071; 40·96428; 68·3214. (5) 17·375; 57; 364. (6) ·8281; 29·42571; 511·25. (7) ·788255. (8) 3·565476190. (9) ·5384645; ·741354. (10) ·6; ·57142857. (11) ·40972; 77·5. (12) 19·3125; 2·125.

XCIV.—(1) 2·733; 3·539; 27·228. (2) 7·918; 235·762; 228·331. (3) 14·887; 628·229; 5·667. (4) 1·581; 274·812; 2·04. (5) 2·367; 8·561; 17·339. (6) 8·118; 37·113; 11·467.

XCV.—(1) £19 17s. $3\frac{1}{2}$d.; £25 7s. 5d.; £176 2s. $8\frac{1}{2}$d. (2) £17 10s. $6\frac{3}{4}$d.; £1 1s. $3\frac{1}{2}$d.; £18 7s. $5\frac{1}{4}$d.; £20 19s. $3\frac{3}{4}$d; £8 5s. 5d. (3) £16 4s. $5\frac{1}{4}$d.; £4 2s. $6\frac{3}{4}$d.; £3 1s. $5\frac{3}{4}$d.; £4 2s. $1\frac{1}{2}$d.; £2 1s. $10\frac{1}{2}$. (4) £4 13s. $11\frac{1}{4}$d.; £18 6s. 6d.; £71 8s. $5\frac{1}{4}$d.; £58 7s. $5\frac{1}{2}$d.; £41 12s. 7d. (5) £7 5s. $8\frac{1}{4}$d.; £5 18s. $0\frac{1}{4}$d.; £8 12s. $5\frac{3}{4}$d.; £5834 14s. $5\frac{1}{4}$d.; £309 12s. 11d. (6) £0 17s. $5\frac{1}{2}$d.; £3 19s. $2\frac{1}{2}$d.; $11\frac{1}{2}$d.; £814 15s. $1\frac{1}{2}$d.; £203 17s. $5\frac{1}{4}$d. (7) 19s. 5d.; 1s. 8d.; 1s. $2\frac{1}{2}$d.; £8 13s. 11d.; £23 14s. 5d.; 1s. $5\frac{1}{2}$d.

XCVII.—(1) £17·514574; £27·6669779; £·109·427082. (2) £273·255205; £12·336456; £1·313533. (3) £28·467697; £1·031248; £23·278123. (4) £17·035414; £29813533; £18·037498.

MISCELLANEOUS EXERCISES ON FRACTIONS.

(1) £818 11s. $1\frac{1}{2}$d. (2) 37·886. (3) $1\frac{1}{2}$ hours. (4) 1. (5) $\frac{27}{160}$; ·16875. (6) 11 hours 29 min. 11 sec.; £45 9s. $6\frac{3}{4}$d. (7) ·0571428; ·0010958904. (8) $\frac{272}{371}$; $\frac{320}{567}$; $1\frac{159}{161}$; $1\frac{97}{400}$; $2\frac{34}{77}$. (9) 6s. $9\frac{1}{7}$d. (10) 3s. $1\frac{1}{4}$d. + $\frac{11}{21}$ of a farthing. (11) 19s. 8d. (12) ·02880 and 125000; ·020412 and 2603·5714285. (13) £1000. (15) ·67142857. (16) £2 13 $7\frac{1}{4}$d. $\frac{3}{7}$, and £48 2s. $11\frac{1}{4}$d. (17) 1st $\frac{1}{3}$, 2nd $\frac{22}{67}$, 3rd $\frac{23}{70}$, 4th $\frac{114}{347}$; $3\frac{2152}{15199}$, $3\frac{1}{7}$, $3\frac{15}{106}$, $3\frac{16}{113}$, $3\frac{791}{5586}$. (18) £354 10s. $3\frac{3}{4}$d. $\frac{9}{20}$. (19) 360. (20) ·368, or $\frac{46}{125}$. (21) 11 and 1. (22) 43·039783 ft.

6·1575164 yds., and 89·03894378 miles. (23) 3·997 in. 1·049 ft. 3·526 miles. (24) £107·3006 ; 355·3. (25) a. 1684·164 ; 23982·05 ; 1739·283. b. 15944·096 ; 710·96375 ; 455·8416. c. 368·7121 ; £897 13s. 9$\frac{1}{4}$d. ; £222 12s. 7$\frac{3}{4}$d. d. £115 7s. 5d. ; £97493 15s. 4d. ; £42 14s. 9$\frac{3}{4}$d. (26) £728. (27) 13$\frac{1}{23}$. (28) 3·61904. (29) $\frac{582}{1253}$; $\frac{671}{1253}$. (30) 6296·41 ; 9189·41 ; 27863·907 ; 1470·63. (31) 3·81024. (32) 25·7069 days. (33) 8·6648125, and 51·988875, and 207·9555, and 3050·014. (34) 98·0835625, and 461816, and 5399·922363. (35) 14s. 3$\frac{2921}{25119}$d., and £86 13s. 7$\frac{43}{25119}$d., and £42 9s. 0$\frac{20228}{25119}$d. (36) £25 3s. 2$\frac{1}{4}$d. (37) As 17952 is to 29920. (38) 58 acres 1 rood 22 poles 1 yd. 0 ft. 32$\frac{56}{59}$ in. (39) £594. (40) 52$\frac{1}{2}$ acres worth £150 11s. 6$\frac{3}{4}$d. (41) £80 9s. 2d., and 4372 francs. (42) 145 lbs. 6 oz. gold, 78 lbs. 6 oz. 6 dwt. silver, 67 lbs. 1 oz. 12 dwt. copper. (43) 776000 years. (44) 1161·33272576, and 4185·3832775, and 5·93956859375. (45) In the 1st case 9515, in the 2nd 11418, and in the 3rd 11893$\frac{3}{4}$. (46) At 6 mils he would lose £1·007 ; at 7 mils he would gain £3·021. (47) Present receipts £66 1s. 8d. ; receipts at the decimal rates £79 6s. 0d.

XCIX.—(1) 20. (2) 7. (3) 204$\frac{17}{57}$. (4) 1653. (5) 3597$\frac{2}{15}$. (6) 24$\frac{1}{4}$. (7) 16640. (8) 450. (9) 250. (10) 4147$\frac{19}{31}$. (11) 172. (12) 85.

CI.—(1) 2 : 11 ; 25 : 48 ; 11 : 126 ; 7 : 15. (2) 117 : 296 ; 27 : 1624 ; 55 : 132 ; 1 : 9. (3) 49 : 31 ; 107 : 117 ; 54 : 101 ; 11 : 125.

CIII.—(1) 185$\frac{597}{914}$ yds. (2) £8131 15s. 3$\frac{1}{2}$d. (3) £1127 19s. 8d. (4) 9 lbs. 12$\frac{4}{5}$ oz. (5) 123$\frac{37}{81}$. (6) 400 and 1400. (7) £4 14s. 1$\frac{1}{2}$d. ; £4 4s. 8$\frac{1}{2}$d. ; £3 15s. 3$\frac{1}{2}$d. ; £3 5s. 10$\frac{1}{2}$d. (8) £2 1s. 9$\frac{3}{4}$d. $\frac{1}{25}$. (9) 4·95 miles, and 121·828 yds. (10) 5$\frac{3}{4}$d. $\frac{23}{191}$. (11) 1s. 1$\frac{1}{2}$d. + $\frac{2}{71}$. (12) 70, and 2$\frac{3}{16}$. (13) £88 0s. 4$\frac{5}{8}$d. (14) 11$\frac{3}{4}$d. + $\frac{1}{7}$. (15) 7s. 11$\frac{3}{4}$d. (16) 88$\frac{1}{6}$. (17) As 847 : 900. (18) 555$\frac{1}{4}$. (19) £52 18s. (20) 97 lbs. 12 oz. (21) 2s. 1$\frac{1}{4}$d. (22) 101$\frac{49}{51}$. (23) 1 hour 48 min. 28 sec. (24) 9 acres 0 roods 24$\frac{77}{200}$ poles. (25) 6s. 1$\frac{1}{2}$d. $\frac{18}{37}$; £18 0s. 5$\frac{1}{4}$d. $\frac{3}{37}$. (26) £834 15s. 1$\frac{11}{80}$d. (27) £9846 5s. 10$\frac{3}{4}$d. $\frac{17}{51}$. (28) £168. (29) 21·$\dot{3}$. (30) 57·448. (31) 160·81 yds. (32) £86907 13s. 1d. (33) £80 16s. 7$\frac{43}{271}$d. (34) 88·267 ft. (35) £9 19s. 11$\frac{1}{4}$d. $\frac{52}{73}$. (36) £745. (37) 8467·093.

CV—(1) 32$\frac{32}{35}$ yds. (2) 17$\frac{1}{5}$ hours. (3) 180 men.

(4) $28\frac{4}{7}$ horses. (5) $11\frac{629}{646}$ months. (6) $16\frac{96}{119}$ days. (7) 125. (8) 20. (9) 200 days. (10) 89. (11) £$613161\frac{9}{31}$. (12) £132. (13) £165 3s. $8\frac{1}{4}$d. + $\frac{16825}{23326}$ quadrants.

CVI.—(1) £17 17s. $7\frac{1}{5}$d. (2) £276 12s. $3\frac{3}{5}$d. (3) £47 8s. $8\frac{4}{5}$d. (4) £87 9s. $3\frac{3}{10}$d. (5) £394 16s. $0\frac{9}{10}$d. (6) £144 6s. $4\frac{1}{2}$d. (7) £175 5s. $4\frac{888}{925}$d. (8) £1880 6s. $9\frac{1}{4}$d. (9) £1003 5s. 6d. (10) £78 9s. 7d. (11) £844 4s. $3\frac{1}{2}$d. (12) £85 12s. $5\frac{3}{4}$d. (13) £519 18s. $0\frac{1}{4}$d. (14) £646 10s. $6\frac{1}{4}$d. (15) £296 14s. $3\frac{3}{4}$d. (16) £51 18s. 1d. (17) £3699 14s. 0d. (18) £2148 11s. 3d. (19) £3526 6s. $11\frac{1}{2}$d. (20) 25 years 225 + days. (21) 16 years 168 + days. (22) 3 years 126 + days. (23) 7 years 218 + days. (24) £4 0s. 9d. + (25) £3 10s. $10\frac{1}{2}$d. + (26) £4 5s. $0\frac{1}{4}$d. + (27) £2 13s. $1\frac{1}{4}$d. + (28) £860. (29) £27400. (30) £2228 11s. $5\frac{1}{7}$d. (31) £1744 10s. $10\frac{1}{2}$d. (32) $6\frac{128}{187}$ years. (33) £6666 13s. 4d. (34) £$4\frac{151}{156}$. (35) 18 years $165\frac{5}{42}$ days. (36) £2826 8s. $5\frac{15}{16}$d.

CVII.—(1) £62 8s. $7\frac{17}{25}$d.; decimally, £62·432. (2) £101 12s. $9\frac{1}{2}$d.; decimally, £101·640. (3) £149 11s. $5\frac{1}{4}$d.; decimally, £149·572. (4) 90·839. (5) £154·5. (6) £713·380. (7) £129·254. (8) £205·892. (9) £404·943. (10) £349·499. (11) £216·784. (12) £191·129. (13) £170·714.

CVIII.—(1) £679 4s. $10\frac{46}{53}$d. (2) £1020. (3) £844 8s. $10\frac{2}{3}$d. (4) 5s. $8\frac{32}{41}$d. (5) £2 4s. $0\frac{51}{80}$d. (6) £1 16s. $10\frac{286}{407}$d. (7) £558 0s. $11\frac{29}{41}$d. (8) £3802 5s. $7\frac{139}{263}$d. (9) a. $11\frac{1}{2}$d., and 1s. $6\frac{1}{4}$d., and 3s. $0\frac{1}{4}$d. b. 3s. $8\frac{3}{4}$d., and £1 0s. $9\frac{3}{4}$d., and 2d., or £·0079. c. £·0028, or about $\frac{3}{4}$d., and $\frac{28514}{26780005}$ of a penny, and 1s. $5\frac{1}{4}$d.

CIX.—(1) £7140 16s. $2\frac{2}{5}$d. (2) £124 6s. 4d. (3) $87\frac{352}{1579}$. (4) £62 1s. 6d. (5) $91\frac{7}{40}$. (6) £5 12s. $8\frac{1}{4}$d. (7) £19 0s. $7\frac{1}{4}$d. (8) £7352 15s. $5\frac{3}{4}$d. (9) £1026 11s. 3d. (10) £27 2s. 6d. (11) $104\frac{1}{8}$. (12) £118 4s. $6\frac{1}{4}$d. (13) £841 12s. $10\frac{1}{2}$d. (14) $97\frac{426479}{1347343}$.

CX.—(1) 110 in geography, 90 in grammar, 30 cannot read, and 10 in algebra. (2) £2072 14s. $6\frac{6}{11}$d. (3) In the first, $617\frac{1}{2}$; and in the second, $314\frac{1}{2}$. (4) 2473·507 ft. nitrogen; 763·4968 ft. oxygen; 36·9962 ft. carbon. (5) $14\frac{22}{27}$ passengers, $55\frac{5}{9}$ rail-servants, and $29\frac{17}{27}$ trespassers. (6) ·0000358. (7) In the first interval, $14\frac{239398}{16364898}$; in the second, $12\frac{3909136}{18658372}$. (8) $20\frac{607630}{2043781}$. (9) Of the first, 42592·2588; of the second, 81808·8102; and the rest, 14516·931. (10) In the public, $79\frac{325749}{1407569}$; and in the private,

$91\frac{47909}{175226}$. (11) $11\frac{13643601}{179237600}$. (12) £173 0s. $4\frac{1}{2}$d. (13) £2240 10s. $1\frac{1}{2}$d.+ (14) £2360·51. (15) Cost price = 3s. $1\frac{3}{4}$d., and gain = £7 8s. $1\frac{7}{8}$d. per cent. (16) $8\frac{148}{369}$. (17) Total gain = £1 19s. 4d., and gain per cent. = $21\frac{83}{277}$. (18) £449 15s. $4\frac{8}{43}$d. (19) 2s. $11\frac{2707}{8600}$d. (20) £869 9s. $2\frac{91}{100}$d.

MISCELLANEOUS EXERCISES IN PROPORTION.

(1) 15. (2) £807 5s. 11d. (3) £2 3s. $6\frac{1}{4}$d. (4) 7200 soldiers. (5) £2 0s. 9d. (6) 2s. $4\frac{1}{2}$d. (7) 972658·575; 14·1426. (8) £5 12s. 1d. (9) £232 5s. 7d. (10) $10\frac{1}{4}$d. (11) £752 10s. 7d. (12) $10\frac{1}{4}$d. (13) £$116\frac{2}{3}$. (14) £$6\frac{2}{3}$. (15) £402 10s. (16) £16 9s. $5\frac{1}{2}$d. (17) £125 7s. $10\frac{1}{4}$d. (18) 4 cwt. 2 qr. $17\frac{97}{103}$ lbs. (19) £70 19s. $6\frac{1}{2}$d. (20) 12 hours. (21) 6s. $1\frac{1}{23}$d. (22) £2 15s. $6\frac{3}{4}$d.; £1388 17s. $9\frac{1}{4}$d. (23) £960 12s. 8d. (24) £12 6s. in the pound; £172 17s. $2\frac{1}{4}$d. (25) £3 2s. $4\frac{1}{2}$d. (26) £589 6s. $2\frac{1}{4}$d. $\frac{4}{5}$. (27) £821 18s. $9\frac{1}{4}$d. (28) £1 15s., and $8\frac{1}{3}$ per cent. (29) £67 2s. $10\frac{3}{4}$d. (30) £3 2s. $5\frac{1}{4}$d. (31) A = £17 14s. $11\frac{1}{71}$d; B = £17 5s. $0\frac{3}{4}$d. $\frac{27}{71}$. (32) 18 men. (33) £19 14s. $3\frac{1}{4}$d. $\frac{5}{7}$. (34) 20·2166025 minutes. (35) $11\frac{20930}{25871}$ years. (36) $30\frac{10}{13}$ years. (37) $22\frac{2}{9}$ per cent. (38) 54; 81; 108; 135; 162. (39) Ought to pay more by 9s. $6\frac{1}{4}$d. $\frac{1}{7}$. (40) £124 10s. $2\frac{1}{4}$d. $\frac{8}{11}$. (41) 160. (42) As 40 : 41; £123 10s. 3d. (43) 3 miles, and $3\frac{1}{4}$ miles per hour. (44) £$1388\frac{8}{9}$. (45) 4 months 1 week $4\frac{1}{13}$ days. (46) £3 12s. $3\frac{1}{2}$d. (47) £406 4s. $9\frac{1}{2}$d. (48) £936 15s. $4\frac{1}{4}$d. + $\frac{181}{427}$.

CXXIII.—(1) 4207, and 835·754, and 78, and 64. (2) 29, and 35·874, and 7538, and 845. (3) 3096, and 213, and 104, and 27·3. (4) 58·06, and 270·2, and ·016, and $\frac{15}{16}$. (5) 2·401, and 317·8, and ·217. (6) 610·2, and 52·615, and 6301·244. (7) 120·79, and 3·14, and ·0769. (8) 3·825, and 4·093, and ·082. (9) ·63245, and ·9826073, and $1\frac{1}{2}$, and 3·111269. (10) 2·5298, and ·8, and ·25298, and ·08, and 25·298. (11) 3·5496, and 8·01248, and 4·27434, and 6·4. (12) ·95452, and 48·6698, and ·3902436, and ·790936. (13) $3\frac{4}{15}$, and $14\frac{37}{40}$. (14) $8\frac{18}{20}$, and $\frac{229}{300}$. (15) $23\frac{24}{35}$, and $20\frac{8}{25}$. (16) $7\frac{14}{31}$, and $4\frac{7}{11}$. (17) 30·0202. (18) 714.

CXXVIII.—(1) 32, and 10·5, and 628. (2) 319, and 4283·77, and 1·72. (3) 1·09, and 16·1, and 51·4. (4) 2009, and 5493·612. (5) 1·077, and 130·011, and ·7631. (6) 1375, and *6281*. (7) 7·092, and 602·8. (8) 5·172, and 37·24. (9) $\frac{14}{16}$, *and* $3\frac{3}{25}$. (10) $\frac{22}{30}$, and $\frac{40}{45}$. (11) $2\frac{13}{15}$, and $6\frac{15}{32}$. (12) a. 90·03,

and 83·03, and 18·45. b. ·843, and ·879, and ·4028, and 3·001. c. 2·593, and 2·636, and 7·988, and 3·976. (13) 1472.

CXXIX.—(1) 45 ft. 3′ 10″ 10‴ 8⁗, and 6 ft. 2′ 1″ 5‴ 5⁗. (2) 355 ft. 1′ 8″ 3‴ 9⁗, and 311 ft. 5′ 2″ 8‴. (3) 1620 ft. 5′ 7″ 11‴, and 383 ft. 0′ 6″ 8‴ 3⁗. (4) 104 ft. 1′ 5″ 3‴ 5⁗ 3′′′′′, and 32·30593 ft. (5) 11·94275 yds., and 14652·97245 ft. (6) $4100\frac{3}{16}$, and $6488\frac{17}{32}$. (7) 87·885 ft. (8) 10·147 ft. (9) 10·26596. (10) 332 ft. 3′ 9″ 6‴ 4⁗ 6′′′′′. (11) 23 ft. 11′ 4″ 2‴ 2⁗. (12) 8 ft. 9′ 9″. (13) 16 ft. 7′. (14) 14 ft. 4′. (15) $45\frac{1}{2}$. (16) 14 yds. 2 ft. $9\frac{3}{7}$′. (17) 34 ft. 4′ 6″ and cost = £1 $1\frac{1}{144}$s. (18) 512, and cost = £51 4s. (19) 785 ft. 9′ 4″. (20) 6′ 4″. (21) £5 17s. $11\frac{5}{48}$d. (22) £11 1s. $2\frac{1}{4}$d. (23) 199 yds. $1\frac{59}{108}$ ft. (24) 11550 : 11421. (25) 40351250 : 50424201. (26) 692111 : 705174. (27) 702 square ft. (28) £2287 14s. 2d. (29) $11\frac{591}{1022}$ ft. (30) £2 13s. $7\frac{49}{72}$d.

CXXX.—16·522, and 18·357. (2) 25·825. (3) 24·392, and 138·971. (4) 1 ft. 9·768′. (5) 10 ft. (6) 19·798, and 29·132, and 46·669. (7) 17·677, and 40·729, and 60·386. (8) 24·624675. (9) 70·5 yds. (10) 506·835. (11) The square of a side, or the square itself. (12) As 62500 : 856381, and as 793881 : 856381. (13) 45·131. (14) 143·443. (15) 71·299, and 4·219.

CXXXI.—(1) 53·40703, and 168·389234, and 775·97273, and 34·243331. (2) 49·019, and 66·399, and 1291·699. (3) 7912·032 miles. (4) 34·377. (5) 3 ft. 3·3719 in. (6) 86° 25′ 44·4″. (7) 280·112 ft. (8) 30·6305. (9) 39·92437. (10) 394 ft. (11) $14\frac{29}{114}$ ft. (12) 32° 13′ 13″.

CXXXII.—(1) 153·93791, and 201·06176, and 254·46879. (2) 4417·8609375, and 5089·572149375, and 40·60198769. (3) 3·101 ft. (4) 124·12 yds. (5) 289 : 974, and as 324 : 974, and as 361 : 974. (6) 5326·237224. (7) 889·8582 ft. (8) 15·22367 ft. (9) 30·7035 ft. (10) 389·93 acres. (11) 25·395 acres. (12) £16 14s. $11\frac{1}{4}$d., and £60 15s. $5\frac{1}{4}$d. (13) 17s. 10d.

CXXXIII.—(1) 6708 ft. 1′ 7″. (2) 1391 ft. 1′ 10″ 6‴, and cost £23 3s. $8\frac{1}{2}$d. nearly. (3) 57375 oz. (4) £531 14s. $6\frac{2}{3}$d. (5) 118800. (6) 3 tons 16 cwt. 1 qr. 27 lbs. (7) 3 tons 5 cwt. 2 qrs. 4 lbs. 10 oz. (8) £3 19s. $3\frac{51}{64}$d. (9) £4 1s. $6\frac{3}{4}$d., and weighs 8 cwt. 2 qrs. 4 lbs. 4 oz.

CXXXIV.—(1) 5038531·3104, and 8379·072. (2) 897649·696 and 2085631807436·53640811 miles. (3) 1795858·536223375 (4) 75729·740625. (5) 1876·136521875. (6) 4199·12281588 (7) 1·5216725. (8) 2 ft. 8′. (9) 14·85 ft. (10) 9199·076026304587 (11) 45 ft. 3′ 10″ 0‴ 2⁗ 11·796′′′′′. (12) 335·879968945.

CXXXV.—(1) 71. (2) 234. (3) 40. (4) 45. (5) 40 (6) 182. (7) 39. (8) 80. (9) 97. (10) 47. (11) (12) 64. (13) 15. (14) 26½. (16) 26¼.

CXXXVI.—(1) 5; 7; 9; 11; 13; 15; 17; 19; 21; 23; 25 27; 29; 31; 33; 35. (2) 15; 22; 29. (3) 8; 11; 14 17; 20; 23; 26; 29; 32; 35; 38; 41; 44; 47; 50; 5 (4) 72; 138; 204; 270; 336; 402; 468; 534.

CXXXVII.—(1) 54, and 57, and 70, and 119. (2) 51, and 8 and 67, and 73. (3) 54, and 60·5, and 9·83. (4) 1, and $\frac{16}{18}$ and $\frac{554}{775}$. (5) $5\frac{21}{80}$, and $31\frac{21}{216}$ and 27. b. (1) 19, and 18, an 27⅓. (2) 5½, and 9⅓, and 22⅔. (3) 14¾, and 9, and 29½. (4) 57 and 8·9, and 4·67.

CXXXVIII.—507, and 39700. (2) 558, and 625. (3) 85 and 18791. (4) 94½, and 175. (5) 288¾, and 897. (6) 85 and 2420. (7) 13$\frac{4}{7}$, and 0. (8) 115, and 45. (9) 109 (10) £8 16s. (11) 21 days. (12) 12502497. (13) £ 7s. 9d. (14) 531·3 ft.

CXXXIX.—(1) 576, and 3249, and 8649. (2) 1875. (3) 11239 (4) 171125. (5) 2187·5. (6) 3468. (7) 5776.

CXLI.—(1) 49152. (2) 4096. (3) 4374. (4) 179 (5) $\frac{3}{512}$. (6). 81920. (7) 3000000. (8) $\frac{2}{243}$. (9) 35429 (10) $\frac{3}{128}$.

CXLII.—(1) 54, and 203, and 18. (2) 60, and 90, and 14 (3) 6, and 40·124, and 984·69.

CXLIII.—(1) 5461, and $39\frac{9}{16}$. (2) $15\frac{6487}{65536}$, and $4\frac{553}{131}$ (3) 109225, and $166\frac{59715}{78125}$.

CXLIV.—(1) £3342 6s. 11¾d. (2) £482 11s. 1d. (3) £11 17s. 2¼d. (4) £13299 16s. 4¾d. (6) £9 5s. 11d.

CXLV.—(1) £2118 11s. 6d. (2) £2715 8s. 4¼d. (3) £4 4s. 0¾d. (4) £2628 15s. 2¼d. (5) £1790 13s. 9 (6) 1208 9s. 0½d.

J. & W. RIDER, Printers, 14, Bartholomew Close, London.